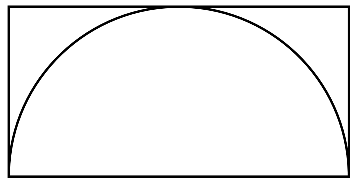
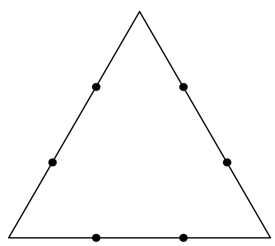
1. A semicircle of radius 2 is inscribed inside of a rectangle, as shown in the diagram below. The diameter of the semicircle coincides with the bottom side of the rectangle, and the semicircle is tangent to the rectangle at all points of intersection. Compute the length of the diagonal of the rectangle.



Answer: $\mathbf{2\sqrt{5}}$

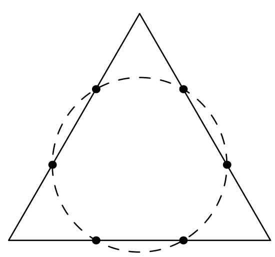
Solution: Two sides of the rectangle are length 4 as they coincide with the diameter of the circle. The other two sides are length 2 because they are the same length as the radius of the circle. Thus, by the Pythagorean Theorem, the length of the diagonal is $\sqrt{2^2+4^2} = \boxed{2\sqrt{5}}$.

2. Consider an equilateral triangle with side length 9. Each side is divided into 3 equal segments by 2 points, for a total of 6 points. Compute the area of the circle passing through these 6 points.



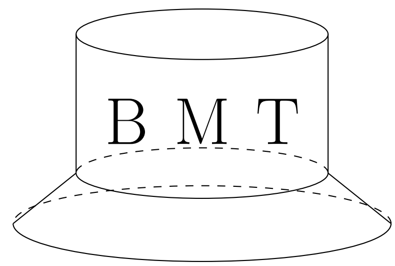
Answer: $\boldsymbol{9\pi}$

Solution:



First, we draw in the circle, as shown in the diagram. The hexagon created by the six points is regular because all of the segments have length $9/3 = 3$ and all of the equilateral triangle's angles are 60°. Furthermore, the diameter of the circle is the longest diagonal of the hexagon, and the longest diagonal of a regular hexagon is twice the length of its side length, and thus is 6. Therefore, the area of the circle must be $3^2\pi = \boxed{9\pi}$.

3. Jingyuan is designing a bucket hat for BMT merchandise. The hat has the shape of a cylinder on top of a truncated cone, as shown in the diagram below. The cylinder has radius 9 and height 12. The truncated cone has base radius 15 and height 4, and its top radius is the same as the cylinder's radius. Compute the total volume of this bucket hat.



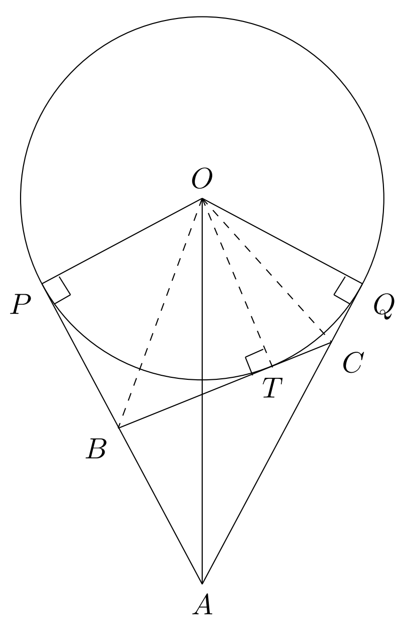
Answer: 1560π

Solution: Recall that the volume of a cone with height h and radius r is $\frac{\pi h r^2}{3}$. The truncated cone is equal to the difference between a cone with radius 15 and height $4+x$ and a cone with radius 9 and height x for some value of x. Using similar triangles, we have that $\frac{x}{9} = \frac{x+4}{15}$. Solving gives $x = 6$. Thus, the volume of the larger cone is $\frac{(10)(15)^2\pi}{3} = 750\pi$ and the volume of the smaller cone is $\frac{(6)(9)^2\pi}{3} = 162\pi$. Thus, the volume of the truncated cone is $750\pi - 162\pi = 588\pi$. Next, the volume of the cylinder is $r^2 \cdot h \cdot \pi = 9^2 \cdot 12 \cdot \pi = 972\pi$. Thus, the total volume of the bucket hat is $588\pi + 972\pi = \boxed{1560\pi}$.

4. Let ω be a circle with center O and radius 8, and let A be a point such that $AO = 17$. Let P and Q be points on ω such that line segments $\overline{AP}$ and $\overline{AQ}$ are tangent to ω. Let B and C be points chosen on $\overline{AP}$ and $\overline{AQ}$, respectively, such that $\overline{BC}$ is also tangent to ω. Compute the perimeter of triangle $\triangle ABC$.

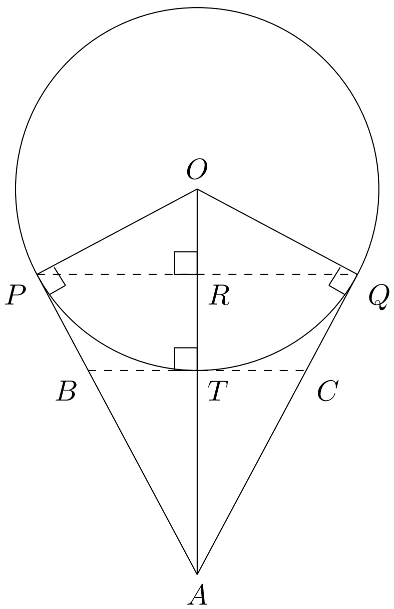
Answer: 30

Solution 1:



Let $\overline{BC}$ be tangent to ω at T. Connecting $\overline{OT}$ gives us $OP = OT = OQ = 8$. By the Pythagorean Theorem, we get $QA = \sqrt{17^2 - 8^2} = 15$. Since $OP = OT$ and $\angle OPB = \angle OTB = 90^\circ$, $\triangle OBP \cong \triangle OBT$. Thus, $PB = BT$. With the same idea, we get $\triangle OCT \cong \triangle OCQ$, and from this, $CT = CQ$. Therefore, the perimeter of $\triangle ABC = AB + BC + AC = AB + BT + TC + AC = BA + BP + CQ + AC = AP + AQ = 15 + 15 = \boxed{30}$.

Solution 2:



An alternative solution can be to select $\overline{BC}$ such that $\overline{BC} \parallel \overline{PQ}$. Let $\overline{BC}$ be tangent to ω at T and $\overline{PQ}$ intersect $\overline{OA}$ at R. Since $\angle OPR + \angle POR = 90^\circ = \angle OPR + \angle APR$, $\angle POR = \angle APR$. In triangles $\triangle POR$ and $\triangle APR$, $\angle POR = \angle APR$, and $\angle ORP = \angle PRA = 90^\circ$, so $\triangle POR \sim \triangle APR$. Also, since $\angle POR = \angle AOP$ and $\angle ORP = \angle OPA = 90^\circ$, $\triangle POR \sim \triangle AOP$. This effectively means $\triangle APR \sim \triangle AOP$. Furthermore, since $\overline{BC} \parallel \overline{PQ}$, $\overline{BC} \perp \overline{AO}$, we also get $\triangle ABT \sim \triangle APR$. By the Pythagorean Theorem, $QA = \sqrt{17^2 - 8^2} = 15$. Since $\triangle APO$ has side lengths in the ratio of $8 : 15 : 17$, so does $\triangle ATB$. So, since $AT = AO - OT = 17 - 8 = 9$, we get $BT = 9 \cdot \frac{8}{15} = \frac{24}{5}$ and $AB = 9 \cdot \frac{17}{15} = \frac{51}{5}$. Therefore, the perimeter of $\triangle ABC = AB + BC + AC = AB + BT + TC + AC = 2 \cdot \left(\frac{24}{5} + \frac{51}{5}\right) = 2 \cdot 15 = \boxed{30}$. We can verify that setting $\overline{BC} \parallel \overline{PQ}$ does not violate any of the constraints of the problem.

5. Triangle $\triangle ABC$ has side lengths $AB = 8$, $BC = 15$, and $CA = 17$. Circles ω_1 and ω_2 are externally tangent to each other and within $\triangle ABC$. The radius of circle ω_2 is four times the radius of circle ω_1. Circle ω_1 is tangent to $\overline{AB}$ and $\overline{BC}$, and circle ω_2 is tangent to $\overline{BC}$ and $\overline{CA}$. Compute the radius of circle ω_1.

   Answer: $\mathbf{\frac{5}{7}}$

   Solution: Let P be the center of ω_1, Q be the center of ω_2, and the radius of the circle ω_1 be r. Divide triangle $\triangle ABC$ into triangles $\triangle AQB$, $\triangle BQC$, and $\triangle CQA$; the key idea is that the heights of these triangles are multiples of r, and we can use the areas of the triangles to find r. Since circle ω_2 is tangent to $\overline{BC}$ and $\overline{CA}$, the heights of triangles $\triangle BQC$ and $\triangle CQA$ to the sides of triangle $\triangle ABC$ are $4r$, so it remains to find the height of triangle $\triangle AQB$. Let X and Y be the points of tangency of $\overline{BC}$ to circles ω_1 and ω_2, respectively. $PXYQ$ is a trapezoid with $\angle PXY = \angle QYX = 90^\circ$, $PX = r$, $QY = 4r$, and $PQ = r + 4r = 5r$. Then, by the Pythagorean Theorem, $XY = \sqrt{PQ^2 - (QY - PX)^2} = \sqrt{(5r)^2 - (3r)^2} = 4r$. So, the height of triangle $\triangle AQB$ is $BY = BX + XY = 5r$.

   The areas of the three triangles mentioned above are $\dfrac{1}{2} \cdot 8 \cdot 5r = 20r$, $\dfrac{1}{2} \cdot 15 \cdot 4r = 30r$, and $\dfrac{1}{2} \cdot 17 \cdot 4r = 34r$, and the area of triangle $\triangle ABC$ is $\frac{1}{2} \cdot 8 \cdot 15 = 60$. Therefore, $20r + 30r + 34r =$

$84r = 60$ and $r = \boxed{\dfrac{5}{7}}$.

6. In triangle $\triangle ABC$, let M be the midpoint of $\overline{AC}$. Extend $\overline{BM}$ such that it intersects the circumcircle of $\triangle ABC$ at a point X not equal to B. Let O be the center of the circumcircle of $\triangle ABC$. Given that $BM = 4MX$ and $\angle ABC = 45°$, compute $\sin(\angle BOX)$.

   Answer: $\boldsymbol{\frac{5\sqrt{7}}{16}}$

   Solution: Since $\angle ABC = 45°$, we have that $\angle AOC = 90°$. Furthermore, if we let $OA = OB = r$, we can then say $AC = \sqrt{2}r$ by the Pythagorean Theorem, and because M is the midpoint of $\overline{AC}$, $AM = MC = \frac{\sqrt{2}}{2}r$. If we draw the circumcircle of $\triangle ABC$, we can use power of a point to set up the equation
   $$BM \cdot MX = AM \cdot AC.$$
   Given that $BM = 4MX$ from the problem statement, we then get
   $$BM = \sqrt{2}r, MX = \frac{\sqrt{2}}{4}r \Rightarrow BX = \frac{5\sqrt{2}}{4}r.$$
   Now that we have all the side lengths for triangle $\triangle BOX$, we can use law of cosines to get
   $$\left(\frac{5\sqrt{2}}{4}r\right)^2 = r^2 + r^2 - 2r^2\cos(\angle BOX) \Rightarrow \cos(\angle BOX) = -\frac{9}{16}.$$
   Thus, $\sin(\angle BOX) = \sqrt{1 - \left(-\frac{9}{16}\right)^2} = \boxed{\dfrac{5\sqrt{7}}{16}}$.

7. A tetrahedron has three edges of length 2 and three edges of length 4, and one of its faces is an equilateral triangle. Compute the radius of the sphere that is tangent to every edge of this tetrahedron.

   Answer: $\boldsymbol{\frac{3\sqrt{11}}{11}}$

   Solution: We first make the observation that the equilateral triangle face must have a side length of 2. If it were side length 4, the other edges would need to be longer than length 2 in order to converge at a fourth point. So, the tetrahedron is a pyramid with an equilateral triangle base, and we can take advantage of its symmetrical properties. By symmetry, the center of the sphere lies on the altitude to the equilateral triangle base. Let P be the apex of the pyramid, let A be a vertex of the equilateral triangle face, let M be the foot of the altitude of the equilateral triangle from point A, let G be the center of the equilateral triangle base (which is also the foot of the altitude of the pyramid from P onto the equilateral triangle base), let O be the center of the sphere, and let B be the foot of the altitude in $\triangle AOP$ from O onto $\overline{AP}$. All of these points are on the same cross-section, and we are given the lengths $PA = 4$, $AM = \sqrt{2^2 - 1^2} = \sqrt{3}$, $PM = \sqrt{4^2 - 1^2} = \sqrt{15}$. Moreover, B is the closest point to O on $\overline{AP}$ and M is the closest point to O on the edge of the equilateral triangle opposite A, so $OB = OM = r$. Since AM is a median, $AG = \frac{2}{3} \cdot \sqrt{3} = \frac{2}{\sqrt{3}}$ and $MG = \frac{1}{3} \cdot \sqrt{3} = \frac{1}{\sqrt{3}}$, and by the Pythagorean Theorem, we know that $OG = \sqrt{r^2 - \frac{1}{3}}$ and $PG = \sqrt{(\sqrt{15})^2 - \frac{1}{3}} = \sqrt{\frac{44}{3}}$, so $OP = \sqrt{\frac{44}{3}} - \sqrt{r^2 - \frac{1}{3}}$. Since $\triangle OBP \sim \triangle AGP$, we have $\frac{OP}{OB} = \frac{AP}{AG}$, or
   $$\frac{\sqrt{\frac{44}{3}} - \sqrt{r^2 - \frac{1}{3}}}{r} = 2\sqrt{3}.$$

We now solve for r. Multiplying both sides by $r\sqrt{3}$ gives $\sqrt{44} - \sqrt{3r^2-1} = 6r$, and squaring both sides gives:

$$\begin{aligned} 44 + (3r^2 - 1) - 2\sqrt{44(3r^2-1)} &= 36r^2 \\ \Rightarrow 2\sqrt{132r^2 - 44} &= -33r^2 + 43. \end{aligned}$$

To simplify things, let $s = 33r^2$, so that $2\sqrt{4s-44} = -s + 43$. Squaring both sides again gives:

$$\begin{aligned} 16s - 176 &= s^2 - 86s + 43^2 \\ \Rightarrow s^2 - 102s + 2025 &= 0. \end{aligned}$$

Observing that $2025 = 45^2 = 3^4 \cdot 5^2$, we can quickly run through its factors to find the factorization $(s-27)(s-75) = 0$, so $s = 27$ or $s = 75$. Plugging these values of s back into $2\sqrt{4s-44} = -s + 43$ reveals that $s = 75$ is extraneous, so $s = 27$. Thus, $33r^2 = 27$, and since r is positive, $r = \sqrt{\frac{27}{33}} = \boxed{\frac{3\sqrt{11}}{11}}$.

8. A circle intersects equilateral triangle $\triangle XYZ$ at A, B, C, D, E, and F such that points X, A, B, Y, C, D, Z, E, and F lie on the equilateral triangle in that order. If $AC^2 + CE^2 + EA^2 = 1900$ and $BD^2 + DF^2 + FB^2 = 2092$, compute the positive difference between the areas of triangles $\triangle ACE$ and $\triangle BDF$.

Answer: $16\sqrt{3}$

Solution: Since triangle $\triangle XYZ$ is an equilateral triangle as given in the problem, let $XY = YZ = ZX = s$. Additionally, let $AX = a$, $BY = b$, $CY = c$, $DZ = d$, $EZ = e$, and $FX = f$.

By Power of a Point from points X, Y, and Z respectively:

$$\begin{aligned} a(s-b) &= f(s-e) \\ c(s-d) &= b(s-a) \\ e(s-f) &= d(s-c). \end{aligned}$$

Adding all three equations together yields

$$\begin{aligned} s(a+c+e) - (ab+cd+ef) &= s(b+d+f) - (ab+cd+ef) \\ a+c+e &= b+d+f. \end{aligned}$$

For the first sum of squared lengths equation, we use the law of cosines with the $60°$ angles:

$$\begin{aligned} 1900 &= AC^2 + CE^2 + EA^2 \\ &= (s-a)^2 - (s-a)c + c^2 + (s-c)^2 - (s-c)e + e^2 + (s-e)^2 - (s-e)a + a^2 \\ &= 2(a^2+c^2+e^2) + (ac+ce+ea) - 3s(a+c+e) + 3s^2 \\ &= 2(a+c+e)^2 - 3(ac+ce+ea) - 3s(a+c+e) + 3s^2. \end{aligned}$$

Similarly, for the second sum of squared lengths equation, we use the law of cosines with the $60°$ angles in the same way to get:

$$2092 = BD^2 + DF^2 + FB^2 = 2(b+d+f)^2 - 3(bd+df+fb) - 3s(b+d+f) + 3s^2.$$

Since $a+c+e=b+d+f$, subtracting the two equations gives

$$192 = BD^2 + DF^2 + FB^2 - (AC^2 + CE^2 + EA^2) = -3(bd + df + fb - ac - ce - ea)$$

$$\Rightarrow ac + ce + ea - bd - df - fb = 64.$$

To find the difference between the areas of triangles $\triangle ACE$ and $\triangle BDF$, we split the areas into multiple triangles and use the sine area formula with the 60° angles:

$$\begin{aligned}[ACE] - [BDF] &= ([XYZ] - [EXA] - [AYC] - [CZE]) \\ &\quad - ([XYZ] - [FXB] - [BYD] - [DZF]) \\ &= \frac{\sqrt{3}}{4} \cdot ((s^2 - c(s-a) - e(s-c) - a(s-e)) \\ &\quad - (s^2 - b(s-d) - d(s-f) - f(s-b))) \\ &= \frac{\sqrt{3}}{4} \cdot (s(b+d+f-a-c-e) \\ &\quad + (ac + ce + ea - bd - df - fb)) \\ &= \frac{\sqrt{3}}{4} \cdot (ac + ce + ea - bd - df - fb) \\ &= \frac{\sqrt{3}}{4} \cdot 64 \\ &= \boxed{16\sqrt{3}}.\end{aligned}$$

9. Let triangle $\triangle ABC$ be acute, and let point M be the midpoint of $\overline{BC}$. Let E be on line segment $\overline{AB}$ such that $\overline{AE} \perp \overline{EC}$. Then, suppose T is a point on the other side of $\overleftrightarrow{BC}$ as A is such that $\angle BTM = \angle ABC$ and $\angle TCA = \angle BMT$. If $AT = 14$, $AM = 9$, and $\frac{AE}{AC} = \frac{2}{7}$, compute BC.

Answer: $\mathbf{6\sqrt{5}}$

Solution: First, note that

$$\begin{aligned}\angle ABT + \angle ACT &= (\angle ABC + \angle CBT) + \angle ACT \\ &= (\angle ABC + \angle MBT) + \angle ACT \\ &= \angle BTM + \angle MBT + \angle BMT \\ &= \pi,\end{aligned}$$

so quadrilateral $ABTC$ is cyclic. Thus, this means that $\angle BTM = \angle ABC = \angle ATC$, so $\overline{AT}$ is a T-symmedian of $\triangle TBC$. Therefore, we also know that $\overline{AT}$ is an A-symmedian.

Consider an inversion of some positive radius r about A. Then, B, C, and T will project to collinear points B', C', and T'. Moreover, we have that $AB \cdot AB' = AC \cdot AC' \Rightarrow \frac{AB}{AC'} = \frac{AC}{AB'}$. Thus, by SAS similarity, we have $\triangle ABC \sim AC'B'$. Next, we have $\angle BAM = \angle TAC = \angle T'AC'$, so therefore T' is the midpoint of $\overline{B'C'}$. Thus, if we define A' to be the point at infinity on line $\overline{B'C'}$ (the inversion of A about itself) then the cross-ratio $(A', T'; B', C') = \frac{B'T'}{C'T'} = -1$, meaning also that $(A, T; B, C) = -1$.

Now, we claim that $AM \cdot AT = AB \cdot AC$. One way to show this is the following: consider our points in the complex plane, with A representing 0 and b, c, and t representing points B, C, and

T. Because $\angle ABT + \angle TCA = \pi$, we have:

$$\begin{aligned}
\left|\frac{b-t}{b} \cdot \frac{c}{c-t}\right| &= 1 \\
\frac{b-t}{b} \cdot \frac{c}{c-t} &= e^{-\pi} \\
\frac{bc-ct}{bc-bt} &= -1 \\
\frac{bt-ct}{bc-bt} &= -2 \\
bt-ct &= -2(bc-bt) \\
-bt-ct &= -2bc \\
\frac{(b+c)t}{2} &= bc \\
\left|\frac{b+c}{2}\right| \cdot |t| &= |b| \cdot |c|
\end{aligned}$$

Since M corresponds to $\frac{b+c}{2}$ in the complex plane, we have that $AM \cdot AT = AB \cdot AC$.

Lastly, by Stewart's Theorem, we have that

$$\begin{aligned}
AB^2 \cdot \frac{BC}{2} + AC^2 \cdot \frac{BC}{2} &= \left(\frac{BC}{2}\right)^2 \cdot BC + AM^2 \cdot BC \\
\frac{1}{2}AB^2 + \frac{1}{2}AC^2 &= \left(\frac{BC}{2}\right)^2 + AM^2 \\
AB^2 + AC^2 &= \frac{BC^2}{2} + 2AM^2,
\end{aligned}$$

and by Law of Cosines, we have that

$$\begin{aligned}
BC^2 &= AB^2 + AC^2 - 2AB \cdot BC \cdot \cos\angle BAC \\
BC^2 &= \frac{BC^2}{2} + 2AM^2 - 2(AM \cdot AT) \cdot \frac{AE}{AC} \\
\frac{BC^2}{2} &= 2 \cdot 9^2 - 2(9 \cdot 14) \cdot \left(\frac{2}{7}\right) \\
BC &= \boxed{6\sqrt{5}},
\end{aligned}$$

as desired.

10. Let triangle $\triangle ABC$ have circumcenter O and circumradius r, and let ω be the circumcircle of triangle $\triangle BOC$. Let F be the intersection of $\overleftrightarrow{AO}$ and ω not equal to O. Let E be on line $\overleftrightarrow{AB}$ such that $\overline{EF} \perp \overline{AE}$, and let G be on line $\overleftrightarrow{AC}$ such that $\overline{GF} \perp \overline{AG}$. If $AC = \frac{65}{63}$, $BC = \frac{24}{13}r$, and $AB = \frac{126}{65}r$, compute $AF \cdot EG$.

Answer: $\frac{156}{25}$ or 6.24

Solution: Let the circumcircle of triangle $\triangle ABC$ be ω_1. Based on the problem statement, there are two valid constructions: either $\angle ACB$ is an acute angle, or $\angle ACB$ is an obtuse angle. It turns out that both configurations achieve the same unique answer. This can be seen from the following argument: consider our circle with center O and radius r with point A on it. From the problem statement, we know that $AB = \frac{126}{65}r$, so construct another circle, ω_2, with center A and radius $\frac{126}{65}r$. Point B must lie on both the first and second circle. These two circles have exactly two intersections, because the radius of the circle with center A is less than the diameter of the circle with center O. However, through congruent triangles, it must be the case that the two intersection points are reflections of each other across $\overline{AO}$, so without loss of generality, we can pick point B as our second point. Then, construct a circle with center B and radius $\frac{24}{13}r$, and notice that this circle intersects ω_2 at two points, C_1 and C_2. Therefore, we can choose either C_1 or C_2 as our third point, C, in our triangle. One of these points will make our triangle acute, and the other one will make our triangle obtuse. To see this, we first notice that A, B, C_1, and C_2 all lie on circle ω_1. Therefore, $\angle AC_1B + \angle AC_2B = 180°$, and since $\overline{AB}$ is not the diameter of ω_1, one of $\angle ACB$ and $\angle AC'B$ is acute, and the other is obtuse.

For readers' better understanding, we have created the following Geogebra diagrams for reference:

- Acute case: `https://www.geogebra.org/calculator/trptfjzt`
- Obtuse case: `https://www.geogebra.org/calculator/ruj5dxnz`

We will consider the case with acute triangle $\triangle ABC$, although the obtuse case follows similarly. Drop an altitude from point A onto $\overline{BC}$ and label this point D. Let $\angle ABC = \alpha$ and $\angle OBC = \beta$. By angle chasing, $\angle OBA = \angle OAB = \alpha - \beta$. Furthermore, $\angle BOC = 180° - \angle OBC - \angle OCB = 180° - 2\beta$, so

$$\angle BAC = 90° - \beta, \angle ACB = 180° - \alpha - (90° - \beta) = 90° - \alpha + \beta$$

and

$$\angle DAC = 180° - 90° - (90° - \alpha + \beta) = \alpha - \beta, \angle OAD = 90° + \beta - 2\alpha.$$

Thus, $\triangle ACD \sim \triangle AFE, \triangle ABD \sim \triangle AFG, \frac{AC}{AF} = \frac{AD}{AE}$, and $\frac{AB}{AD} = \frac{AF}{AG}$. From this, we get that $\triangle ACF \sim \triangle ADE$ and $\triangle ADG \sim \triangle ABF$ so $\angle DGA = \angle BFA$. Since $BFCO$ is a cyclic quadrilateral on circle ω, then

$$\angle DGA = \angle BFA = \angle BFO = \angle BCO = \beta.$$

Thus, $\angle EBF = \angle BAF + \angle BFA = (\alpha - \beta) + \beta = \alpha$. Notice that

$$\angle BAD = \angle BAO + \angle OAD = (\alpha - \beta) + (90° + \beta - 2\alpha) = 90° - \alpha$$

and

$$\angle ABD = 180° - 90° - (90° - \alpha) = \alpha = \angle EBF,$$

meaning that $\triangle DBE \sim \triangle ABF \sim \triangle ADG$. Thus, $\angle BED = \angle BFA = \beta$. Since $\angle AED + \angle EAC = \beta + (90° - \beta) = 90°$, then $\overline{ED} \perp \overleftrightarrow{AG}$, and we also know $\overline{FG} \perp \overleftrightarrow{AG}$ and $\overline{ED} \parallel \overline{FG}$. Similarly, $\angle DGA + \angle EAC = \beta + (90° - \beta) = 90°$, $\overline{DG} \perp \overleftrightarrow{AE}$, $\overline{EF} \perp \overleftrightarrow{AE}$, and $\overline{DG} \parallel \overline{EF}$. We conclude that $EDGF$ is a parallelogram.

Since $\angle AEF + \angle AGF = 90° + 90° = 180°$, the quadrilateral $AEFG$ is cyclic. Applying Ptolemy's Theorem on cyclic quadrilateral $AEFG$, we get

$$AF \cdot EG = AE \cdot FG + AG \cdot EF = AE \cdot ED + AG \cdot DG.$$

Now, in triangle $\triangle OBC$, we have $\cos\beta = \frac{BC}{2r} = \frac{12}{13}$. Similarly, $\cos(\alpha-\beta) = \frac{AB}{2r} = \frac{63}{65}$, and $\cos\alpha = \frac{4}{5}$. By Law of Sines, $\frac{AD}{\sin\beta} = \frac{ED}{\sin(90^\circ-\alpha)} = \frac{ED}{\cos\alpha}$ and $\frac{AD}{\sin\beta} = \frac{DG}{\sin(\alpha-\beta)}$. Thus,

$$\begin{aligned}[ADG] &= \frac{1}{2}\cdot AD\cdot DG\cdot\sin(180^\circ-\alpha)\\ &= \frac{1}{2}\cdot AD\cdot DG\cdot\sin\alpha\\ &= \frac{AD^2\cdot\sin(\alpha-\beta)\cdot\sin\alpha}{2\sin\beta}\end{aligned}$$

and

$$\begin{aligned}[ADE] &= \frac{1}{2}\cdot AD\cdot ED\cdot\sin(90^\circ+\alpha-\beta)\\ &= \frac{1}{2}\cdot AD\cdot ED\cdot\cos(\alpha-\beta)\\ &= \frac{AD^2\cdot\cos(\alpha-\beta)\cdot\cos\alpha}{2\sin\beta}.\end{aligned}$$

Additionally, $[ADG]$, $[ADE]$, and AD can be expressed as $[ADG] = \frac{AG\cdot DG\cdot\sin\beta}{2}$, $[ADE] = \frac{AE\cdot ED\cdot\sin\beta}{2}$, and $AD = AC\cdot\cos(\alpha-\beta) = \frac{65}{63}\cdot\frac{63}{65} = 1$.

Plugging in all the values, $AE\cdot ED = AD^2\cdot\frac{4\cdot63\cdot13}{5^4}$ and $AD\cdot DG = AD^2\cdot\frac{48\cdot13}{5^4}$. Thus,

$$AF\cdot EG = AD^2\cdot\left(\frac{4\cdot63\cdot13}{5^4}+\frac{48\cdot13}{5^4}\right) = AD^2\cdot\frac{3900}{625} = \boxed{\frac{156}{25}}.$$