1. Compute the three-digit number that satisfies the following properties:

   - The hundreds digit and ones digit are the same, but the tens digit is different.
   - The number is divisible by 9.
   - When the number is divided by 5, the remainder is 1.

   Answer: 171

   Solution: For our number to be one more than a multiple of 5, it must end in a 1 or a 6. Since the hundreds digit is the same as the ones digit, the number is either of the form $\underline{1}\,\underline{A}\,\underline{1}$ or $\underline{6}\,\underline{B}\,\underline{6}$. The divisibility rule for 9 gives that $1 + A + 1$ and $6 + B + 6$ would have to be multiples of 9, so $A = 7$ and $B = 6$. This gives 171 and 666 as candidate numbers, but the tens digit must be different from the other digits, so the only three-digit number satisfying all conditions is $\boxed{171}$.

2. Three people, Pranav, Sumith, and Jacklyn, are attending a fair. Every time a person enters or exits, the groundskeeper writes their name down in chronological order. If each person enters and exits the fairgrounds exactly once, in how many ways can the groundskeeper write down their names?

   Answer: 90

   Solution: Let A, B, C represent the events of the three friends entering and let X, Y, Z represent the events of friends exiting. Then the number we're looking for is the number of permutations of $ABCXYZ$ where A comes before X, B comes before Y, and C comes before Z. This is $\frac{6!}{2^3}$, as we take any permutation and then swap $(A, X), (B, Y)$, or (C, Z) if needed. There are 2^3 possible combinations of swaps we may need to do, and exactly one of these swaps is valid. So, for every $2^3 = 8$ permutations, there is exactly one valid permutation. Thus, the answer is $\frac{6!}{2^3} = \boxed{90}$.

3. Find the number of positive integers n less than 10000 such that there are more 4's in the digits of $n + 1$ than in the digits of n.

   Answer: 1111

   Solution: Note that increasing n by one can only affect more than the last digit if the last digit is a 9, in which case we also essentially add one to the tens digit. Thus, without carrying, the only way to increase in the number of 4s in the base-10 expansion is for the last digit to be a 3. However, if we carry once, we need the units digit to be a 9 and the tens digit to be a 3, and so on. It follows that, for a number k, there are more 4s in the base-10 representation of $k + 1$ than k if and only if k ends in $39 \ldots 9$, where the number of 9s is a nonnegative integer.

   Now, for every $n \in \{0, 1, 2, 3\}$, suppose k has the form

   $$d_n d_{n-1} \cdots d_1 3 \underbrace{9 \ldots 9}_{4-n-1}.$$

   There are 10^n positive integers of this form, as the d_i can be any digit. Summing over all n, we the total number of positive integers is $1 + 10 + 100 + 1000 = \boxed{1111}$.

4. Given positive integers $a \geq 2$ and k, let $m_a(k)$ denote the remainder when k is divided by a. Compute the number of positive integers, n, less than 500 such that $m_2(m_5(m_{11}(n))) = 1$.

   Answer: 182

Solution: We slowly unwrap the condition. Our condition that $m_2(m_5(m_{11}(k))) = 1$ is equivalent to the condition that $m_5(m_{11}(k)) \equiv 1 \pmod 2$, which is equivalent to

$$m_{11}(k) \pmod 5 \in \{1, 3\},$$

which is equivalent to $k \pmod{11} \in \{1, 3, 6, 8\}$. Among all the integers $\{1, 2, \ldots, 11\}$, and indeed for every 11 consecutive positive integers, we see that $\frac{4}{11}$ have $k \pmod{11} \in \{1, 3, 6, 8\}$. Therefore, the total number of positive integers between 1 and 495 satisfying this condition is $\frac{4}{11} \cdot 495 = 180$. We also need to consider 496 and 498, so the final answer is $180 + 2 = \boxed{182}$.

5. Kait rolls a fair 6-sided die until she rolls a 6. If she rolls a 6 on the Nth roll, she then rolls the die N more times. What is the probability that she rolls a 6 during these next N times?

Answer: $\mathbf{\frac{6}{11}}$

Solution: Note that, for every positive integer i, the probability that Kait first rolls a 6 on the ith roll is $\left(\frac{5}{6}\right)^{i-1}\left(\frac{1}{6}\right)$. Then, given that she rolls a 6 on the ith roll, the probability that she doesn't roll a 6 on the next i rolls is $\left(\frac{5}{6}\right)^{i}$. Therefore, the probability that she rolls a 6 on the ith roll and then not on the next i rolls is $\left(\frac{5}{6}\right)^{2i-1}\left(\frac{1}{6}\right)$. Thus, the final probability is $\left(\frac{5}{6}\right)^{1}\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^{3}\left(\frac{1}{6}\right) + \cdots$. This is a geometric sequence with initial term $\frac{5}{36}$ and common ratio of $\left(\frac{5}{6}\right)^2 = \frac{25}{36}$. Therefore, the probability that she *won't* roll a 6 during these next N rolls is $\frac{\frac{5}{36}}{1-\frac{25}{36}} = \frac{5}{11}$. We want the probability that the complementary event occurs, or $1 - \frac{5}{11} = \boxed{\frac{6}{11}}$.

6. Let N be the number of positive integers x less than $210 \cdot 2023$ such that

$$\operatorname{lcm}(\gcd(x, 1734), \gcd(x + 17, x + 1732))$$

divides 2023. Compute the sum of the prime factors of N with multiplicity. (For example, if $S = 75 = 3^1 \cdot 5^2$, then the answer is $1 \cdot 3 + 2 \cdot 5 = 13$).

Answer: 51

Solution: First, we clean up the expression slightly to make it a bit easier for us. In particular, by the Euclidean Algorithm,

$$\gcd(x + 17, x + 1732) = \gcd(x + 17, (x + 1732) - (x + 17)) = \gcd(x + 17, 1715).$$

So, our new problem involves when $\operatorname{lcm}(\gcd(x, 1734), \gcd(x + 17, 1715))$ divides 2023. In order for this to be satisfied, we simply need that $\gcd(x, 1734)$ and $\gcd(x + 17, 1715)$ both divide 2023.

Now, it would be good to know what the prime factorizations of 1715 and 1734 are. Dividing 1715 by 5 leaves 343, so $1715 = 5 \cdot 7^3$. Additionally, since $34 = 2 \cdot 17$, then 17 divides 1734, and in particular $1734 = 17 \cdot 102$, which is $2 \cdot 3 \cdot 17^2$. Alternatively, we see that 2 and 3 both divide 1734, and then $\frac{1734}{6} = 289 = 17^2$.

Using the factorizations of $1715, 1734$, and 2023, we can create the conditions required on x. For example, because 7^2 divides 1715, then it cannot divide $x + 17$ or else $7^2 | \gcd(x + 17, 1715)$, meaning that $2023 = 7 \cdot 17^2$ cannot divide it. In this way, our conditions are that $5 \nmid (x + 17)$, that $7^2 \nmid (x + 17)$, that $2 \nmid x$, and that $3 \nmid x$. Notice that $5 \nmid (x + 17)$ is equivalent to $x \not\equiv 3 \pmod 5$, and the other three conditions can be written similarly. So, by the Chinese Remainder Theorem, out of every $5 \cdot 7^2 \cdot 2 \cdot 3$ consecutive numbers, exactly $4 \cdot (7^2 - 1) \cdot 1 \cdot 2 = 2^7 \cdot 3$ of these numbers will satisfy all of these conditions. Therefore, out of the first $210 \cdot 2023$ positive numbers, since $210 \cdot 2023 = (5 \cdot 7^2 \cdot 2 \cdot 3) \cdot 17^2$, then $(2^7 \cdot 3) \cdot 17^2$ of these smallest positive numbers will satisfy all four of our conditions. Moreover, note that $7^2 \mid (210 \cdot 2023)$, implying that S (which excludes the number $210 \cdot 2023$) is also $2^7 \cdot 3 \cdot 17^2$. Thus, the answer is $2 \cdot 7 + 3 + 17 \cdot 2 = \boxed{51}$.

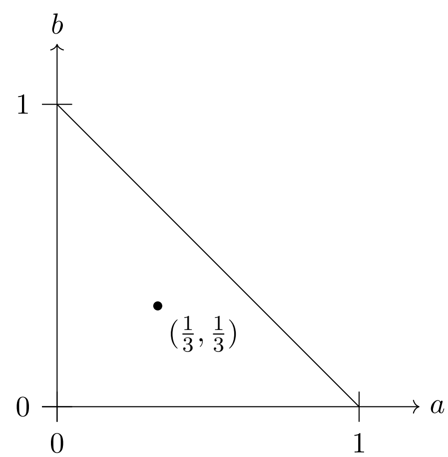
7. Maria and Skyler have a square-shaped cookie with a side length of 1 inch. They split the cookie by choosing two points on distinct sides of the cookie uniformly at random and cutting across the line segment formed by connecting the two points. If Maria always gets the larger piece, what is the expected amount of extra cookie in Maria's piece compared to Skyler's, in square inches?

   Answer: $\mathbf{\frac{11}{18}}$

   Solution: We split this into two cases, when the selected points are on adjacent sides and when they are on opposite sides.

   The probability of the first case is $\frac{2}{3}$, and the expected value of the difference is $|(1 - \frac{ab}{2}) - \frac{ab}{2}|$, where a and b are independent random variables distributed across $[0, 1]$, because of the random distribution across the sides of the square. The smaller piece will always be a triangle with side lengths a and b. The expected value of the area of this smaller triangle is $\mathbb{E}\left[\frac{1}{2}ab\right] = \frac{1}{2}\mathbb{E}[a] \cdot \mathbb{E}[b] = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. So, the expected value of the other piece is $1 - \frac{1}{8} = \frac{7}{8}$, and the expected value of the difference is $\frac{7}{8} - \frac{1}{8} = \frac{3}{4}$.

   The probability of the second case is $\frac{1}{3}$, and the expected value of the difference is $\left|\left(1 - \frac{a+b}{2}\right) - \frac{a+b}{2}\right|$, because points on opposite sides would determine two trapezoids, with a and b defined in the same way as the prior case. This value simplifies to $|1 - (a + b)|$. This expression needs to be evaluated differently, because $(a + b)$ may be greater than 1. The expected difference, however, should be symmetrical across both cases. So, we only need to find the expected difference given that $(a + b) < 1$. We can solve this with geometric probability. Graphing, we get something similar to the following figure:



   In reality, this is a 3D graph of a pyramid with the difference in areas (i.e. $1 - (a + b)$) as the height in a dimension above this plane. To find the expected difference here, we can consider that the distribution is uniform, and the centroid of the graphed triangle is the point with the average of all differences. The centroid of this triangle is at $\left(\frac{1}{2} \cdot \frac{2}{3}, \frac{1}{2} \cdot \frac{2}{3}\right) = \left(\frac{1}{3}, \frac{1}{3}\right)$. So, the expected value of the difference in this case is $|1 - \frac{2}{3}| = \frac{1}{3}$. An alternate way to calculate this is to notice that the volume is $\frac{1}{6}$, but we must multiply this by 2 because we're assuming $(a+b) < 1$, which happens with probability $\frac{1}{2}$.

Thus, the expected difference in area is $\frac{2}{3} \cdot \frac{3}{4} + \frac{1}{3} \cdot \frac{1}{3} = \boxed{\frac{11}{18}}$.

8. Define a family of functions $S_k(n)$ for positive integers n and k by the following two rules:

$$S_0(n) = 1,$$
$$S_k(n) = \sum_{d|n} dS_{k-1}(d).$$

Compute the remainder when $S_{30}(30)$ is divided by 1001.

Answer: 232

Solution: The first step is to realize for relatively prime numbers p, q, we have $S_k(pq) = S_k(p)S_k(q)$. We can prove this with induction on k: first, for the base case, we see that $S_0(pq) = 1 = S_0(p)S_0(q)$. Then, using the fact that the divisors of pq correspond one-to-one with pairs of (relatively prime) divisors of p and q, we get the following in the inductive step:

$$\begin{aligned} S_k(pq) &= \sum_{d|pq} dS_{k-1}(d) \\ &= \sum_{d_p|p} \sum_{d_q|q} d_p d_q S_{k-1}(d_p d_q) \\ &= \sum_{d_p|p} \sum_{d_q|q} d_p S_{k-1}(d_p) d_q S_{k-1}(d_q) \\ &= \left(\sum_{d_p|p} d_p S_{k-1}(d_p) \right) \left(\sum_{d_q|q} d_q S_{k-1}(d_q) \right) \\ &= S_k(p) \cdot S_k(q). \end{aligned}$$

So, we want to compute $S_{30}(2)S_{30}(3)S_{30}(5)$. In order to do this, we compute $S_k(p)$ for arbitrary integer k and prime p. After computing the first few values of $S_k(p)$ for $k = 0, 1, 2, \ldots$, we observe a pattern:

$$S_0(p) = 1, S_1(p) = 1 + p, S_2(p) = 1 + p + p^2, \ldots.$$

We claim that

$$S_k(p) = \sum_{i=0}^{k} p^i.$$

We can prove this fairly easily via induction again:

$$S_k(p) = 1S_{k-1}(1) + pS_{k-1}(p) = 1 + \sum_{i=1}^{k} p^i = \sum_{i=0}^{k} p^i,$$

based on the fact that $S_k(1) = 1$, which is fairly simple to see.

Finally, we have the equation $S_{30}(30) = (2^{31} - 1) \cdot \frac{3^{31}-1}{2} \cdot \frac{5^{31}-1}{4}$. We compute this modulo 7, 11, and 13, the prime factors of 1001, making use of Fermat's Little Theorem:

$$S_{30}(30) = (2^{31} - 1) \cdot \frac{3^{31} - 1}{2} \cdot \frac{5^{31} - 1}{4} \equiv (2 - 1) \cdot \frac{3 - 1}{2} \cdot \frac{5 - 1}{4} = 1 \pmod{7},$$

$$S_{30}(30) = (2^{31}-1)\cdot\frac{3^{31}-1}{2}\cdot\frac{5^{31}-1}{4} \equiv (2-1)\cdot\frac{3-1}{2}\cdot\frac{5-1}{4} = 1 \pmod{11}.$$

13 poses a little more of an issue. We have

$$S_{30}(30) = (2^{31}-1)\cdot\frac{3^{31}-1}{2}\cdot\frac{5^{31}-1}{4} \equiv (2^7-1)\cdot\frac{3^7-1}{2}\cdot\frac{5^7-1}{4} \pmod{13}.$$

Now, $2^7 - 1 = 128 - 1 \equiv -3 \pmod{13}$, and we notice $3^3 = 27 \equiv 1 \pmod{13}$ so we get $\frac{3^7-1}{2} = \frac{3-1}{2} = 1$. Then, for the factor corresponding to 5, we get $5^2 = 25 \equiv -1 \pmod{13}$, so we get $\frac{5^7-1}{4} \equiv \frac{-5-1}{4} \equiv (-6)(-3) \equiv 5 \pmod{13}$. Therefore, the total residue modulo 13 is $(-3)(1)(5) \equiv 11 \pmod{13}$.

The last step is combining these residues via Chinese Remainder Theorem. We can easily combine $1 \pmod 7$ and $1 \pmod{11}$ into $1 \pmod{77}$ by inspection. Then notice that 78 is $1 \pmod{77}$ and $0 \pmod{13}$, so we start with -2 which is -2 mod 13 and 77. So, we add three copies of 78 to -2 to get $234 - 2 = \boxed{232}$.

9. Shiori places seven books, numbered from 1 to 7, on a bookshelf in some order. Later, she discovers that she can place two dividers between the books, separating the books into left, middle, and right sections, such that:

- The left section is numbered in increasing order from left to right, or has at most one book.
- The middle section is numbered in decreasing order from left to right, or has at most one book.
- The right section is numbered in increasing order from left to right, or has at most one book.

In how many possible orderings could Shiori have placed the books? For example, $(2,3,5,4,1,6,7)$ and $(2,3,4,1,5,6,7)$ are possible orderings with the partitions $2,3,5|4,1|6,7$ and $2,3,4|1|5,6,7$, but $(5,6,2,4,1,3,7)$ is not.

Answer: 544

Solution: Note that zero- or one-book sections are technically in increasing and decreasing order by definition. We'll refer to orderings of the books with permutations of $\{1,2,3,4,5,6,7\}$. Consider the following naive counting scheme:

- Place each of the elements of $\{1,2,3,4,5,6,7\}$ into one of the sets A, B, or C (where A, B, and C can be empty). This can be done in 3^7 ways.
- Write down the elements of A in increasing order.
- After that, write down the elements of B in decreasing order.
- After that, write down the elements of C in increasing order.
- The resulting string of numbers is a possible ordering, with partitions separating the sets.

This will successfully always give us a valid ordering, but will sometimes return duplicate orderings, which we will need to account for. Let

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$$

be a permutation of $\{1,2,3,4,5,6,7\}$ that is a possible ordering, and suppose that it is not equal to the identity permutation $(1,2,3,4,5,6,7)$. Define $a_0 = 0$ and $a_8 = 8$, so that $(a_0, a_1, \ldots, a_7, a_8)$

has the least term first and the greatest term last. Let i be the least index such that $a_i > a_{i+1}$, and let j be the least index such that $j > i$ and $a_j < a_{j+1}$; in other words, i and j are points of inflection in the sequence. Then, it can be verified that the leftmost partition must be placed immediately to the left or right of a_i, and the rightmost partition must be placed immediately to the left or right of a_j, and that each of these 4 possible placements of partitions works. Therefore, the scheme overcounts all possible orderings, excluding the identity permutation, 4 times.

For the identity permutation $(1, 2, 3, 4, 5, 6, 7)$, the only restriction for the partitions is that the middle section has zero or one books, which give 8 and 7 possible locations for the partitions, respectively, for a total of 15 possible placements of the dividers. So, of the 3^7 tuples of sets (A, B, C), 15 of them correspond to the identity permutation, and the other $3^7 - 15$ tuples correspond to the other possible orderings, which are each counted 4 times, implying that the total number of possible orderings is $\dfrac{3^7 - 15}{4} + 1 = \boxed{544}$.

10. Let α denote the positive real root of the polynomial $x^2 - 3x - 2$. Compute the remainder when $\lfloor \alpha^{1000} \rfloor$ is divided by the prime number 997. Here, $\lfloor r \rfloor$ denotes the greatest integer less than r.

Answer: 970

Solution: For brevity, set $p = 997$. Let β denote the other root of $x^2 - 3x - 2$. Observe

$$|\beta| = \left| \frac{3 - \sqrt{17}}{2} \right| < 1.$$

Also, $\alpha^{p+3} + \beta^{p+3}$ is an integer (for example, one can see this via a Newton's sums argument because $\alpha^{p+3} + \beta^{p+3}$ is a symmetric polynomial in the roots of $x^2 - 3x - 1$), and $0 < \beta^{p+3} < 1$ means that $\alpha^{p+3} + \beta^{p+3} - 1$ is equal to $\lfloor \alpha^{p+3} \rfloor$.

It remains to compute $\alpha^{p+3} + \beta^{p+3} - 1 \pmod p$. Now that everything involved is safely an integer, we acknowledge that we can embed α and β into $\mathbb{F}_p\left[\sqrt{17}\right]$, which is the field conjured by adjoining $\sqrt{17}$ to $\mathbb{F}_p$.

The main claim is that $\alpha^p = \beta$ and $\beta^p = \alpha$ in $\mathbb{F}_p\left[\sqrt{17}\right]$. One way to see this is through abstract algebra: we have that $\alpha, \beta \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$ because $\sqrt{17} \notin \mathbb{F}_p$, so the Frobenius automorphism $x \mapsto x^p$ must swap α and β. However, we can also argue directly. We show $\alpha^p = \beta$ and a symmetric argument establishes $\beta^p = \alpha$. We compute

$$\alpha^p = \left(\frac{3 + \sqrt{17}}{2} \right)^p = \frac{3^p + \left(\sqrt{17}\right)^p}{2^p} = \frac{3 + 17^{(p-1)/2}\sqrt{17}}{2}$$

because $(x + y)^p = x^p + y^p$ for any $x, y \in \mathbb{F}_p\left[\sqrt{17}\right]$. Continuing the computation using Euler's criterion and quadratic reciprocity, we have

$$17^{(p-1)/2} \equiv \left(\frac{17}{p}\right) = \left(\frac{997}{17}\right) = \left(\frac{-6}{17}\right) = \left(\frac{-1}{17}\right)\left(\frac{2}{17}\right)\left(\frac{3}{17}\right) = 1 \cdot 1 \cdot -1 = -1,$$

so we indeed see $\alpha^p = \frac{3 - \sqrt{17}}{2} = \beta$ in $\mathbb{F}_p[\sqrt{17}]$.

We are now ready to compute our answer. We have

$$\alpha^{p+3} + \beta^{p+3} = \beta\alpha^3 + \alpha\beta^3 = \alpha\beta\left((\alpha + \beta)^2 - 2\alpha\beta\right) = -2 \cdot \left(3^2 - 2 \cdot -2\right) = -26.$$

Thus, $\alpha^{p+3} + \beta^{p+3} - 1 \equiv -26 - 1 \equiv -27 \equiv \boxed{970} \pmod{997}$.