**Time limit:** 1 Week.
**Maximum score:** 150 points.
**Instructions:** For this test, you work in teams of up to six to solve a multi-part, proof-oriented question.

Problems that use the words "compute", "list", or "draw" only call for an answer; no explanation or proof is needed. Unless otherwise stated, all other questions require explanation or proof. Answers should be written on sheets of scratch paper, clearly labeled, with every problem subpart *on its own sheet*. If you have multiple pages for a problem, number them and write the total number of pages for the problem (e.g. 1/2, 2/2).

In your solution for a given problem, you may cite the statements of earlier problems (but not later ones) without additional justification, even if you haven't solved them. The problems are ordered by content, NOT DIFFICULTY. It is to your advantage to attempt problems from throughout the test.

Write your team ID number and team name clearly on each sheet. Only submit one set of solutions for the team. Do not turn in any scratch work. This power round is due on September 25, 2020 at 11:59 PM PST. Please follow the instructions at the link below to submit the power round. You may LaTeXyour solutions or computer process them if you wish.

Submission Instructions Link: `https://www.ocf.berkeley.edu/~bmt/django_media/touranent_docs/BMT_2020_Online/Power_Round_Submission_and_Problem_Disputes.pdf`

You may and are encouraged to use outside tools such as Geogebra, Wolfram Alpha or Desmos to solve the power round. We hightly recommend the following link for graphing:

`https://www.geogebra.org/geometry?lang=en`

You may NOT, however, consult with other teams.
If you find any inconsistencies with the Power Round, please email berkeleymt@gmail.com with your concern. We will do our best to update any errata for the Power Round on our website `https://bmt.berkeley.edu`.

## Introduction

The Billiards Masters Tournament (BMT), in an effort to boost its reputation, is about to host its annual crazy billiards tournament and is hiring you to analyze its newest tables. Your job is to determine crazy mathematical properties that BMT can use to help its billiard masters get better at their trade.

Good luck, and have fun!

## Mathematical Billiards

The mathematics of Billiards is essentially the same as the mathematics that physicists use when examining optics. In general, when light hits a mirror, the trajectory of light obeys the **law of reflection**, explained in the figure below. Similarly, when a billiards ball hits the edge of the table, it always bounces off according to this law. This technique has been employed by Billiards Masters for years.

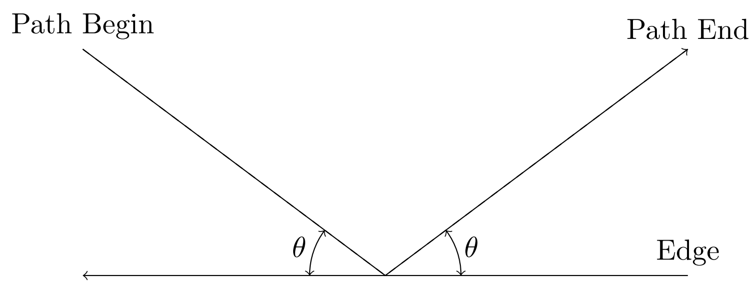



Figure 1: The Law of Reflection: Trajectories once they hit an edge, bounce off of that edge with the same angle they entered at.

A **billiards table** $T$ is a two dimensional bounded polygon with special points called **pockets**. We note that the corners of $T$ are *always* pockets. A curve on $T$ is called a **trajectory** if it travels along a straight line, follows the law of reflection when it hits the edge of the table, and stops when it hits a pocket. The reason for this is that the law of reflection is undefined at corners.

We can think of a trajectory as a function of time $P(t)$ that takes in a non-negative time $t$ and returns a position of a ball on the billiards table after $t$ seconds, assuming that all billiard balls move at a speed of one unit per second.

Since the law of reflection is symmetric, we can also imagine "running the clock in reverse." So we define $P(t)$ for $t < 0$ as the trajectory $\overline{P}(-t)$, where $\overline{P}(t)$ is the trajectory of the ball starting at $P(0)$ and moving in the opposite direction as $P$.

The **starting point** of the ball is therefore $P(0)$. Starting points can be anywhere on the table, including on the edges, as long as they are not pockets.

## Analyzing Billiard Trajectories

**Definition.** We say a billiards trajectory $P(t)$ on a surface is **periodic** if it "loops" back onto itself. That is, there exists a positive constant $C$ such that $P(t+C) = P(t)$ for all $t$. The minimum positive constant $C$ for which this is true is known as the **period** of the trajectory.

**Definition.** A trajectory $P$ is **degenerate** if there exists a $t > 0$ such that $P(t)$ is a pocket.

**Definition.** A trajectory $P_1(t)$ **contains** a trajectory $P_2(t)$ if there exists a positive constant $C$ such that $P_1(t+C) = P_2(t)$. We call $P_1$ a **rewind** of $P_2$. Similarly, $P_2$ is a **fast-forward** of $P_1$.

**Definition.** The **combinatorial period** of a periodic trajectory $T$ on a mathematical billiards table is the size of the following set:

$$\{0 < t \leq C \mid P(t) \text{ is on an edge}\}$$

where $C$ is the period of $T$. In other words, it is the number of times the trajectory hits an edge before repetition.

**Note:** The trajectory can hit the same edge multiple times, and in fact it can hit the same point multiple times. These are counted as distinct.

**Definition.** The **slope** of a trajectory is the initial slope of the the line $\overline{P(0)P(t_1)}$ where $t_1$ is the first time $P$ hits a wall.

**Definition.** An $n$**-billiards table** is a regular $n$-gon with unit side lengths and pockets at all corners. We also implicitly embed the $n$-billiards table on the 2D-plane such that it is oriented such that its bottom-most edge, called the **base**, is on the positive $x$-axis, and the left-most point of the base is the origin.

**Example.** Here is an example of a periodic billiard trajectory on the 4-billiards table (i.e unit square now treated as a billiard table).

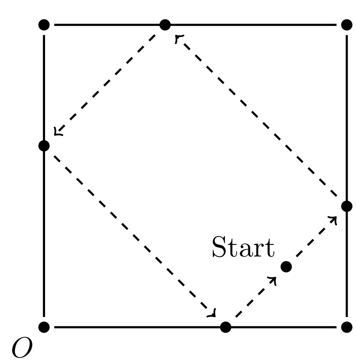



Figure 2: A periodic trajectory (dashed) on the unit square. Starts at $(0.8, 0.2)$ moving at a $45^\circ$ angle with the base.

*Remark.* For the entire power round, whenever angles are specified, they are specified as **counter-clockwise** from the base of the billiards table. For instance, in the figure above, the periodic trajectory is moving at a 45 degree angle from the base of the 4-billiards table.

1. Draw out the following trajectories and determine whether the following trajectories are either periodic or degenerate on the 4-billiards table. If they are periodic, determine their combinatorial period. No proof required.

   (a) **[5]** A trajectory that starts at $(0.2, 0.6)$ moving towards the point $(0.3, 0.9)$.

   (b) **[5]** A trajectory that starts at $\left(\frac{\pi}{4}, \frac{\pi}{5}\right)$ moving diagonally up and to the left (i.e. at a $135°$ angle relative to the base).

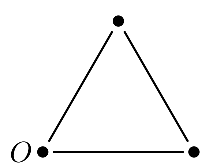
2. **[10]** Show that for any periodic trajectory on any billiards table, its combinatorial period must be greater than 1.

3. (a) **[3]** Show that if trajectory $P_1$ contains $P_2$ then $P_1$ is degenerate if and only if $P_2$ is degenerate.

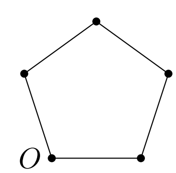
   (b) **[4]** Show that if trajectory $P_1$ contains $P_2$ then $P_1$ is periodic if and only if $P_2$ is periodic.

   (c) **[3]** Show that every periodic trajectory $P$ has a rewind $Q$ such that $Q(0)$ is on the edge of the table.

## Periodic Trajectories on Regular Polygons




(a) The 3-billiards table




(b) The 5-billiards table

Figure 3: The black dots represent pockets and origin is located at the bottome left pocket for each table.

4. (a) **[1]** Draw the periodic trajectory on the 4-billiards table starting at the point $(\frac{1}{2}, 0)$ and moving at an angle of $45°$ from the base. Compute its combinatorial period?

   (b) **[1]** Draw the periodic trajectory on the 3-billiards table starting at the point $(\frac{1}{2}, 0)$ and moving at an angle of $60°$ from the base. Compute its combinatorial period?

   (c) **[1]** Draw the periodic trajectory on the 5-billiards table starting at the point $(\frac{1}{2}, 0)$ and moving at an angle of $72°$ from the base. Compute its combinatorial period?

   (d) **[4]** Consider an $n$-billiards table. Label the pockets of the table $C_0 \dots C_{n-1}$ counter-clockwise from the origin and let the points $P_0 \dots P_{n-1}$ be defined such that $P_i$ is the midpoint of $\overline{C_i C}_{i+1 \pmod n}$. Show that for $i \neq j$, $\angle C_{i+1 \pmod n} P_i P_j$ is $\left(\frac{180(j-i \pmod n)}{n}\right)^{\circ}$.

   **Note**: $C_i C_{i+1 \pmod n}$ means that $i, i+1$ are taken modulo $n$ (i.e $C_0 = C_n$).

   (e) **[3]** Show that for any $n$-billiards table, a trajectory starting at the midpoint of the base and traveling at an angle of $\left(\frac{180k}{n}\right)^{\circ}$ from the base is periodic, where $0 < k < n$ and $k$ is an integer. Determine, with proof, its combinatorial period in terms of $k$ and $n$?

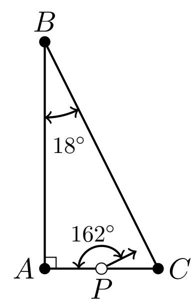
5. (a) **[1]** Draw out the periodic trajectory on the 3-billiards table starting at the point $\left(\frac{1}{2}, 0\right)$ and moving at an angle of $30°$ from the base.

(b) **[1]** Draw out the periodic trajectory on the 5-billiards table starting at the point $\left(\frac{1}{2}, 0\right)$ and moving at an angle of $54°$ from the base.

(c) **[6]** Consider an $n$-billiards table $T$ with $n$ odd. Suppose we label the vertices of $T$ with $A_1, A_2, \ldots, A_n$ starting at the topmost vertex moving clockwise. Let $P$ be the midpoint of the base. Show that there exists a point $Q$ on $\overline{A_1A_2}$ such that $\angle PQA_1$ is a right angle.

(d) **[2]** Show there exists a periodic trajectory of combinatorial period 4 on any $n$-billiards table with $n$ odd.

6. Consider billiards table formed by the right triangle $\triangle ABC$ with pockets at its corners below (not to scale). The base is $\overline{AC}$.




(a) **[4]** Draw the periodic trajectory starting at midpoint $P$ of $\overline{AC}$ and moving at an $18°$ angle relative to segment $\overline{AC}$.

(b) **[6]** Let $T$ be a table constructed from any right triangle $\triangle ABC$ with pockets at its corners. Show that $T$ has a periodic trajectory with combinatorial period 6.

7. This problem is an extension of problem 4. We **highly** recommend doing it before attempting this one.

(a) **[1]** Draw the following trajectory: a periodic trajectory on the 3-billiards table starting at the point $(\frac{3}{4}, 0)$ and moving at an angle of $60°$. Compute its combinatorial period?

(b) **[1]** Draw the following trajectory: a periodic trajectory on the 5-billiards table starting at the point $(\frac{3}{5}, 0)$ and moving at an angle of $72°$. Compute its combinatorial period?

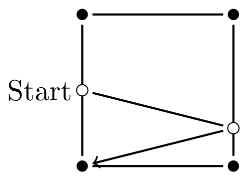
(c) **[6]** Consider an $n$-billiards table. Show that for any point $p$ on the base (not including pockets), any non-degenerate trajectory starting from $p$ moving at an angle of $\left(\frac{180k}{n}\right)^\circ$ is periodic, where $0 < k < n$ and $k$ is an integer.

(d) **[2]** Consider an $n$-billiards table. Show that for any point $p$ in the **interior of the table** (not including edges or pockets), any non-degenerate trajectory starting from $p$ moving at an angle of $\left(\frac{180k}{n}\right)^\circ$ from the base is periodic, where $k$ is any integer.

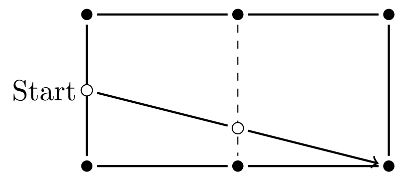
## Unfolding Billiards Tables

One way to analyze trajectories on Billiard tables is through a technique called **unfolding**. Let us illustrate this phenomenon with a simple example. Consider the 4-billiards table trajectory that starts on the midpoint of the left edge reflects off of the right most edge and enters the bottom left pocket. If we were to reflect the shape (or unfold the shape) over the right-most edge, the trajectory would become a straight line!

**Example.** We will show how to unfold a square billiards table into larger rectangular one.



The starting Billiards Table



After reflection across the rightmost edge.

This holds true for any billiards table! In general, once you unfold a billiards table, it becomes another billiards table, where the edge that was reflected over disappears. Similarly, unfolding a periodic trajectory always yields another periodic trajectory and vice versa. We can also unfold multiple times over before deleting the edges.

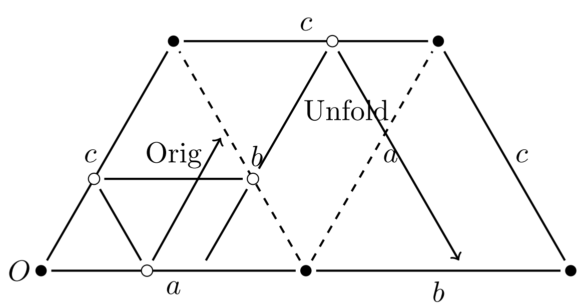
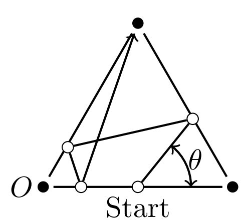
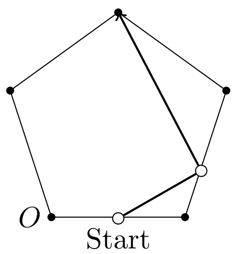



Figure 5: First unfolding over edge $b$ to create tables and then over edge $a$ on the newly reflected shape.

8. Let us apply unfolding to actual billiard paths. Compute the quantities associated with the following degenerate trajectories:




(A): **[5]** The value of $\tan(\theta)$ on the 3-Billiards Table. Note that the trajectory starts at the midpoint of the base. Remember the triangle has unit side lengths.




(B): **[5]** The length of the above trajectory on the 5-Billiards table. Note that the trajectory starts at the midpoint of the base. **Note:** $\cos(108^\circ) = \frac{1-\sqrt{5}}{4}$. Remember the pentagon has unit side lengths!

9. **[10]** Consider table $T$ and edge $e$. Consider the unfolding transformation $U : T \to R$ that reflects $T$ over edge $e$ and then deletes edge $e$. The resulting reflected table that consists of two copies of $T$, with one reflected, is $R$.

   Now consider a periodic path $P$ on table $T$ that has combinatorial period $k$, and hits edge $e$ exactly $n$ times. Determine, with proof, the combinatorial period of $U(P)$ on $R$. You may assume $T$ is a convex polygon with pockets solely at its corners.

We can repeatedly unfold a shape until it becomes a larger, more familiar billiards table. It so turns out that the equilateral triangle unfolds into a regular hexagon, so extending a trajectory on the equilateral triangle, we have:

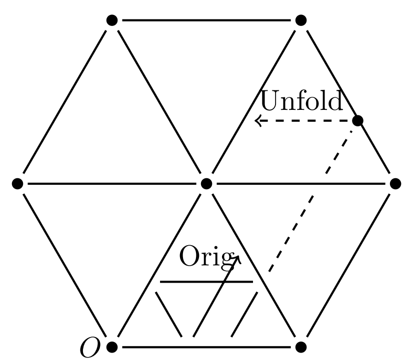



Figure 6: The Unfolded Trajectory of an Equilateral Triangle

10. Demonstrate how to unfold the following billiard tables into their corresponding shapes. Give a diagram that explains how this is done. This need not be rigorous.

    (a) **[3]** A $45^\circ - 45^\circ - 90^\circ$ triangle to a square.

    (b) **[3]** A $36^\circ - 90^\circ - 54^\circ$ triangle to a pentagon.

    (c) **[4]** A $36^\circ - 36^\circ - 108^\circ$ triangle to a 5-pointed star with $72^\circ$ internal angles and $144^\circ$ external angles.

11. Suppose you have a billiards table $T$ that is a $\left(\frac{180}{n}\right)^\circ - 90^\circ - \left(\frac{90n-180}{n}\right)^\circ$ right triangle with pockets at its corners. Orient the triangle such that the side directly opposite the $\left(\frac{180}{n}\right)^\circ$ angle is the base.

(a) [**3**] Show that $T$ can be unfolded into a regular $n$-gon.

(b) [**3**] Show that if $n$ is even, then starting from any point on the base (not including pockets) of $T$, any non-degenerate trajectory moving vertically upward is periodic.

(c) [**4**] Determine the combinatorial period of the above trajectory in terms of $n$ where $n$ is even. Justify your answer.

## Challenge Problems

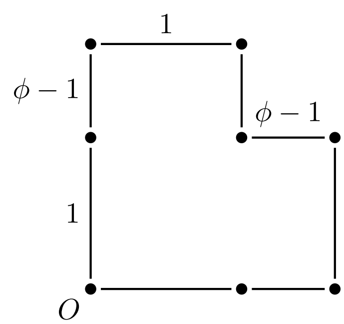
These problems all use techniques and tools built up in the power round as main ideas, but you will have to use your own creativity to finish them off. Good luck!

12. (a) [**5**] Show that a non-degenerate trajectory on the 4-billiards table is periodic if and only if it has a rational or undefined slope.

    (b) [**4**] Suppose the slope of a non-degenerate trajectory on the 4 billiards table is rational and can be written as the reduced fraction $\frac{a}{b}$. Determine, with proof, its combinatorial period (in terms of $a$ and $b$).

    (c) [**1**] Compute the combinatorial period of a non-degenerate trajectory with slope $\frac{2020}{2021}$.

13. The **Golden L** is a billiards table that looks as follows:




[**10**] Show that any periodic trajectory on the golden L has either an undefined slope, or a slope of the form $a + b\phi$ where $a, b \in \mathbb{Q}$ where $\mathbb{Q}$ is the rational numbers and $\phi$ is the golden ratio ($\frac{1+\sqrt{5}}{2}$). As before pockets are black dots. **Hint:** Rewind and track the horizontal and vertical distances separately.

14. [**10**] Let $T$ be a billiards table that is an acute triangle with pockets at its corners. Show that $T$ contains exactly two periodic trajectories of combinatorial period 3. Note that trajectories that are rewinds or fast-forwards of each other are considered the same in this case.

15. For this problem, we will need a couple of tools:

**Definition.** Define a **simple billiards table** be a convex polygon with pockets at its corners.

**Theorem.** *For any simple billiards table, if you are given a line segment that starts at point $P_1$ on $e_1$ and ends at point $P_2$ on $e_2$, and $P_1, P_2$ are not pockets, then there exists an $\epsilon$ such that if you translate $\overline{P_1P_2}$ in a direction perpendicular to the slope of $\overline{P_1P_2}$ by any $t < \epsilon$ and extend/crop it to line segment $\overline{P_1^tP_2^t}$ with $P_1^t$ on $e_1$ and $P_2^t$ on $e_2$, the line $\overline{P_1^tP_2^t}$ never contains a pocket.*

[**10**] Show that for any periodic trajectory with odd combinatorial period $p$ on a simple billiards table, there exists a periodic trajectory with combinatorial period $2p$.