Time limit: 90 minutes.
Instructions: This test contains 25 short answer questions. All answers are positive integers. Only submitted answers will be considered for grading.
No calculators.

1. Justin throws a standard six-sided die three times in a row and notes the number of dots on the top face after each roll. How many different sequences of outcomes could he get?

2. Let m be the answer to this question. What is the value of $2m - 5$?

3. At Zoom University, people's faces appear as circles on a rectangular screen. The radius of one's face is directly proportional to the square root of the area of the screen it is displayed on. Haydn's face has a radius of 2 on a computer screen with area 36. What is the radius of his face on a 16×9 computer screen?

4. Let a, b, and c be integers that satisfy $2a + 3b = 52$, $3b + c = 41$, and $bc = 60$. Find $a + b + c$.

5. A Yule log is shaped like a right cylinder with height 10 and diameter 5. Freya cuts it parallel to its bases into 9 right cylindrical slices. After Freya cut it, the combined surface area of the slices of the Yule log increased by $a\pi$. Compute a.

6. Haydn picks two different integers between 1 and 100, inclusive, uniformly at random. The probability that their product is divisible by 4 can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

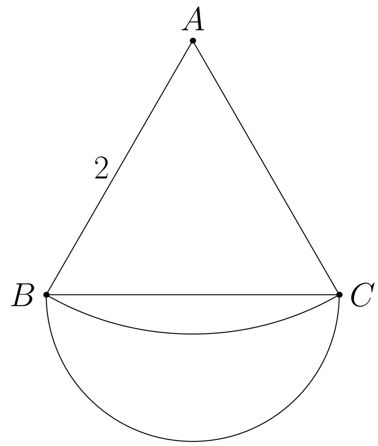
7. A square has coordinates at $(0, 0), (4, 0), (0, 4)$, and $(4, 4)$. Rohith is interested in circles of radius r centered at the point $(1, 2)$. There is a range of radii $a < r < b$ where Rohith's circle intersects the square at exactly 6 points, where a and b are positive real numbers. Then $b - a$ can be written in the form $m + \sqrt{n}$, where m and n are integers. Compute $m + n$.

8. By default, iPhone passcodes consist of four base-10 digits. However, Freya decided to be unconventional and use hexadecimal (base-16) digits instead of base-10 digits! (Recall that $10_{16} = 16_{10}$.) She sets her passcode such that exactly two of the hexadecimal digits are prime. How many possible passcodes could she have set?

9. A circle C with radius 3 has an equilateral triangle inscribed in it. Let D be a circle lying outside the equilateral triangle, tangent to C, and tangent to the equilateral triangle at the midpoint of one of its sides. The radius of D can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

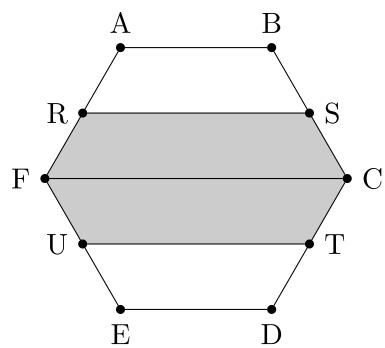
10. Given that p and $p^4 + 34$ are both prime numbers, compute p.

11. Equilateral triangle ABC has side length 2. A semicircle is drawn with diameter $\overline{BC}$ such that it lies outside the triangle, and minor arc $\widehat{BC}$ is drawn so that it is part of a circle centered at A. The area of the "lune" that is inside the semicircle but outside sector ABC can be expressed in the form $\sqrt{p} - \frac{q\pi}{r}$, where p, q, and r are positive integers such that q and r are relatively prime. Compute $p + q + r$.



12. Compute the remainder when 98! is divided by 101.

13. Sheila is making a regular-hexagon-shaped sign with side length 1. Let $ABCDEF$ be the regular hexagon, and let R, S, T and U be the midpoints of FA, BC, CD and EF, respectively. Sheila splits the hexagon into four regions of equal width: trapezoids $ABSR, RSCF, FCTU$, and $UTDE$. She then paints the middle two regions gold. The fraction of the total hexagon that is gold can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.



14. Let B, M, and T be the three roots of the equation $x^3 + 20x^2 - 18x - 19 = 0$. What is the value of $|(B + 1)(M + 1)(T + 1)|$?

15. The graph of the degree 2021 polynomial $P(x)$, which has real coefficients and leading coefficient 1, meets the x-axis at the points $(1, 0), (2, 0), (3, 0), \ldots, (2020, 0)$ and nowhere else. The mean of all possible values of $P(2021)$ can be written in the form $a!/b$, where a and b are positive integers and a is as small as possible. Compute $a + b$.

16. The triangle with side lengths 3, 5, and k has area 6 for two distinct values of k: x and y. Compute $\left|x^2 - y^2\right|$.

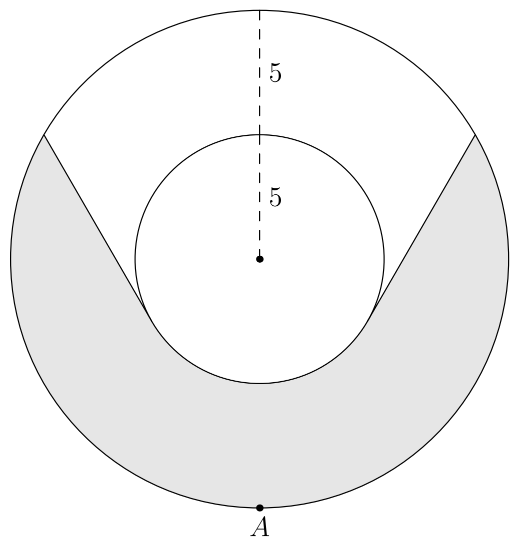
17. Shrek throws 5 balls into 5 empty bins, where each ball's target is chosen uniformly at random. After Shrek throws the balls, the probability that there is exactly one empty bin can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

18. Let x and y be integers between 0 and 5, inclusive. For the system of modular congruences

$$\begin{cases} x + 3y \equiv 1 \pmod{2} \\ 4x + 5y \equiv 2 \pmod{3} \end{cases},$$

find the sum of all distinct possible values of $x + y$.

19. Alice is standing on the circumference of a large circular room of radius 10. There is a circular pillar in the center of the room of radius 5 that blocks Alice's view. The total area in the room Alice can see can be expressed in the form $\frac{m\pi}{n} + p\sqrt{q}$, where m and n are relatively prime positive integers and p and q are integers such that q is square-free. Compute $m + n + p + q$. (Note that the pillar is not included in the total area of the room.)



20. Compute the number of positive integers $n \leq 1890$ such that n leaves an odd remainder when divided by all of 2, 3, 5, and 7.

21. Let P be the probability that the product of 2020 real numbers chosen independently and uniformly at random from the interval $[-1, 2]$ is positive. The value of $2P - 1$ can be written in the form $(\frac{m}{n})^b$, where m, n and b are positive integers such that m and n are relatively prime and b is as large as possible. Compute $m + n + b$.

22. Three lights are placed horizontally on a line on the ceiling. All the lights are initially off. Every second, Neil picks one of the three lights uniformly at random to switch: if it is off, he switches it on; if it is on, he switches it off. When a light is switched, any lights directly to the left or right of that light also get turned on (if they were off) or off (if they were on). The expected number of lights that are on after Neil has flipped switches three times can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

23. Circle Γ has radius 10, center O, and diameter $\overline{AB}$. Point C lies on Γ such that $AC = 12$. Let P be the circumcenter of $\triangle AOC$. Line $\overleftrightarrow{AP}$ intersects Γ at Q, where Q is different from A. Then the value of $\frac{AP}{AQ}$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

24. Let N be the number of non-empty subsets T of $S = \{1, 2, 3, 4, \ldots, 2020\}$ satisfying $\max(T) > 1000$. Compute the largest integer k such that 3^k divides N.

25. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that for all $x, y \in \mathbb{R}^+$, $f(x)f(y) = f(xy) + f\left(\frac{x}{y}\right)$, where $\mathbb{R}^+$ represents the positive real numbers. Given that $f(2) = 3$, compute the last two digits of $f\left(2^{2^{2020}}\right)$.