1. A Yule log is shaped like a right cylinder with height 10 and diameter 5. Freya cuts it parallel to its bases into 9 right cylindrical slices. After Freya cut it, the combined surface area of the slices of the Yule log increased by $a\pi$. Compute a.

   Answer: 100

   Solution: In order to create the 9 slices, Freya makes 8 cuts, each of which is parallel to the bases of the cylinder. Each cut creates two new surfaces, which are circles with diameter 5. The increase in surface area, therefore, is $16\left(\frac{\pi\cdot 5^2}{4}\right) = 100\pi$, and our answer is $\boxed{100}$.

2. Let O be a circle with diameter $AB = 2$. Circles O_1 and O_2 have centers on $\overline{AB}$ such that O is tangent to O_1 at A and to O_2 at B, and O_1 and O_2 are externally tangent to each other. The minimum possible value of the sum of the areas of O_1 and O_2 can be written in the form $\frac{m\pi}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

   Answer: 3

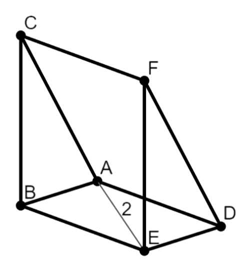
   Solution 1: Let r_i denote the radius of O_i for $i = 1, 2$ and $[O_i]$ denote the area of circle O_i. Since $r_1 + r_2 = 1$, minimizing $[O_1] + [O_2] = \pi(r_1^2 + r_2^2)$ is the same thing as minimizing

   $$\pi(r_1^2 + (1 - r_1)^2) = \pi(r_1^2 + 1 - 2r_1 + r_1^2) = \pi(2r_1^2 - 2r_1 + 1).$$

   Since we know that the minimum of the quadratic ax^2+bx+c occurs at $x = -\dfrac{b}{2a}$, the value of r_1 that minimizes this quantity is $r_1 = -\dfrac{-2}{2(2)} = \dfrac{1}{2}$, so the minimum sum of areas is $\pi(2r_1^2-2r_1+1) = \pi\left(2\left(\dfrac{1}{2}\right)^2 - 2\left(\dfrac{1}{2}\right) + 1\right) = \frac{\pi}{2}$, and our answer is $\boxed{3}$.

   Solution 2: Using the same notation as before, $[O_1] + [O_2] = \pi(r_1^2 + r_2^2) \geq 2\pi r_1 r_2$ by the arithmetic mean-geometric mean inequality, with equality occurring if and only if $r_1 = r_2 = \dfrac{1}{2}$. Thus, the minimum possible area is $2 \cdot \dfrac{\pi}{4} = \frac{\pi}{2}$, and our answer is $\boxed{3}$.

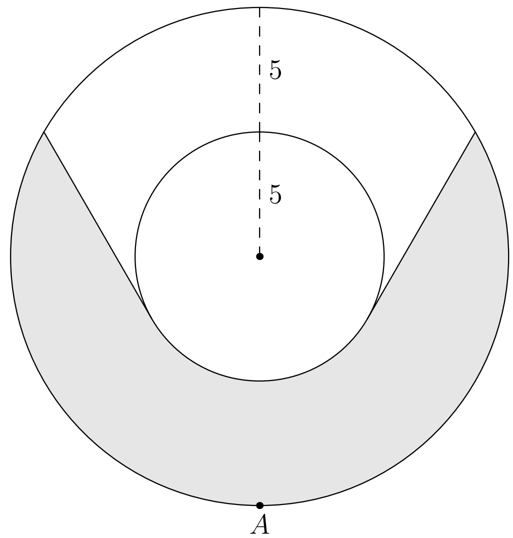
3. Right triangular prism $ABCDEF$ with triangular faces $\triangle ABC$ and $\triangle DEF$ and edges $\overline{AD}, \overline{BE}$, and $\overline{CF}$ has $\angle ABC = 90^\circ$ and $\angle EAB = \angle CAB = 60^\circ$. Given that $AE = 2$, the volume of $ABCDEF$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.



   Answer: 5

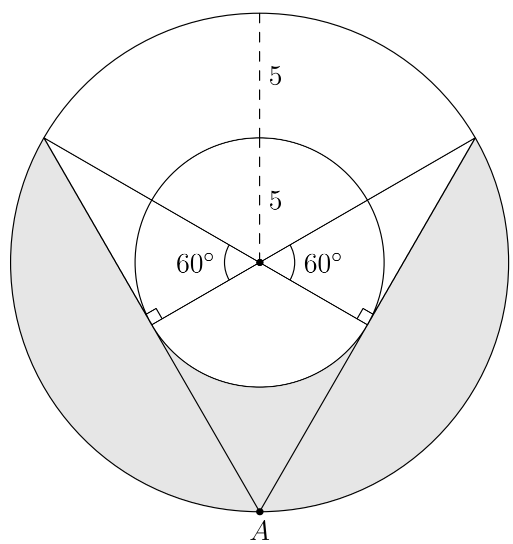
   Solution: The volume of $ABCDEF$ is equal to the area of $\triangle ABC$ multiplied by the height BE. We have that the height is $AE\sin(60^\circ) = \sqrt{3}$ and $BA = AE\cos(60^\circ) = 1$, so $\triangle ABC$ is a 30-60-90 right triangle. Then its area is $\dfrac{\sqrt{3}}{2}$, and the volume of $ABCDEF$ is $\frac{3}{2}$. Our answer, therefore, is $\boxed{5}$.

4. Alice is standing on the circumference of a large circular room of radius 10. There is a circular pillar in the center of the room of radius 5 that blocks Alice's view. The total area in the room Alice can see can be expressed in the form $\frac{m\pi}{n}+p\sqrt{q}$, where m and n are relatively prime positive integers and p and q are integers such that q is square-free. Compute $m+n+p+q$. (Note that the pillar is not included in the total area of the room.)



Answer: 156

Solution:



The region is composed of a 120° sector of the annulus plus two 60° sectors with radius 10, minus two 30-60-90 triangles of side lengths $5, 5\sqrt{3}$, and 10 (see diagram). The area of the annulus sector is $\frac{120}{360}\pi(10^2-5^2)=25\pi$, the total area of the two triangles is $2\cdot\frac{25\sqrt{3}}{2}=25\sqrt{3}$, and the total area of the 60° sectors is $2\cdot\frac{60}{360}\cdot\pi\cdot 10^2=\frac{100\pi}{3}$. Adding and subtracting in the right order gives an area of

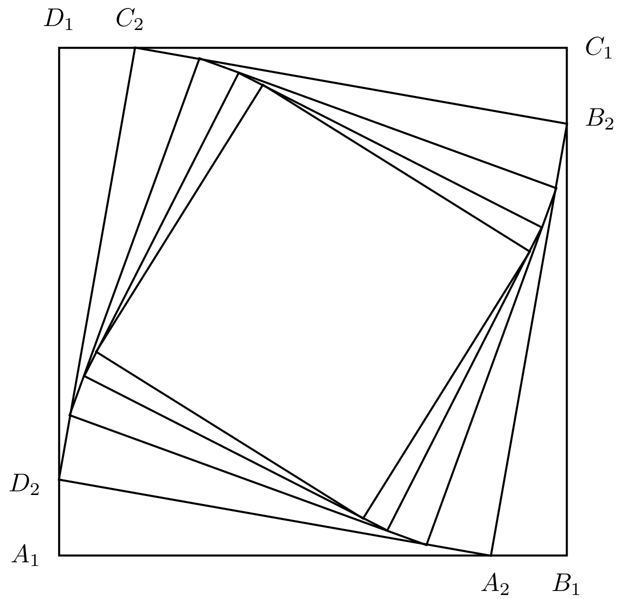
$$25\pi-25\sqrt{3}+\frac{100\pi}{3}=\frac{175\pi}{3}-25\sqrt{3}$$

and thus our final answer is $\boxed{156}$.

5. Let $A_1 = (0,0), B_1 = (1,0), C_1 = (1,1), D_1 = (0,1)$. For all $i > 1$, we recursively define

$$A_i = \frac{1}{2020}(A_{i-1} + 2019B_{i-1})$$
$$B_i = \frac{1}{2020}(B_{i-1} + 2019C_{i-1})$$
$$C_i = \frac{1}{2020}(C_{i-1} + 2019D_{i-1})$$
$$D_i = \frac{1}{2020}(D_{i-1} + 2019A_{i-1}),$$

where all operations are done coordinate-wise.



If $[A_iB_iC_iD_i]$ denotes the area of $A_iB_iC_iD_i$, there are positive integers a, b, and c such that

$$\sum_{i=1}^{\infty}[A_iB_iC_iD_i] = \frac{a^2b}{c},$$

where b is square-free and c is as small as possible. Compute the value of $a + b + c$.

Answer: 3031

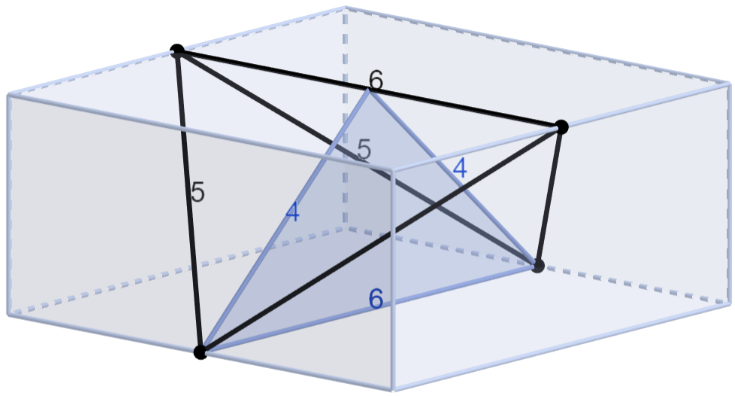
Solution: We note that by symmetry, there is a k such that $[A_iB_iC_iD_i] = k[A_{i-1}B_{i-1}C_{i-1}D_{i-1}]$ for all i. We can see that $1 = [A_1B_1C_1D_1] = [A_2B_2C_2D_2] + 4[A_1A_2D_2] = [A_2B_2C_2D_2] + \frac{4038}{2020^2}$, hence $k = 1 - \frac{2019}{2 \cdot 1010^2}$. Using the geometric series formula, we get

$$\sum_{i=1}^{\infty}[A_iB_iC_iD_i] = \frac{1}{1-k} = \frac{1010^2 \cdot 2}{2019} \implies \boxed{3031}.$$

6. A tetrahedron has four congruent faces, each of which is a triangle with side lengths 6, 5, and 5. If the volume of the tetrahedron is V, compute V^2.

Answer: 252

Solution:

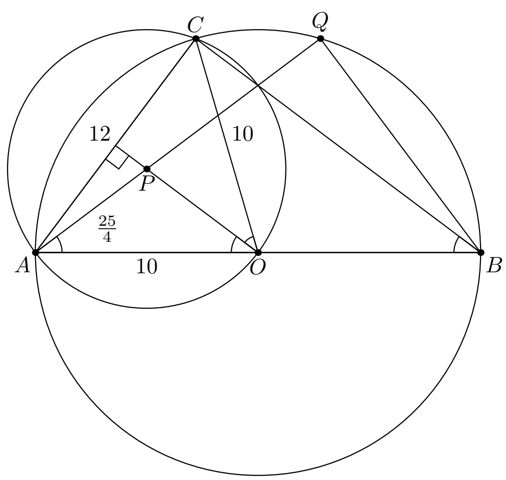


Cut the tetrahedron in half such that the cross section forms a 4-4-6 triangle (see diagram above). Note that the height (from a side of length 4) of this triangle is equal to the height of the tetrahedron. To find the height of this triangle, we can use $A = \frac{bh}{2} \implies h = \frac{2A}{b}$. Since $b = 4$ and $A = 3\sqrt{7}$ (by splitting the 4-4-6 triangle into 2 congruent right triangles), we have that $h = \frac{2A}{b} = \frac{3\sqrt{7}}{2}$. Additionally, by using Heron's or the Pythagorean theorem, the base of our tetrahedron has the same area as one of the 6-5-5 triangles, which has area $\frac{1}{2}(6 \cdot 4) = 12$, so our total volume is $\frac{bh}{3} = 6\sqrt{7}$ and our answer is $\boxed{252}$.

7. Circle Γ has radius 10, center O, and diameter $\overline{AB}$. Point C lies on Γ such that $AC = 12$. Let P be the circumcenter of $\triangle AOC$. Line $\overleftrightarrow{AP}$ intersects Γ at Q, where Q is different from A. Then the value of $\frac{AP}{AQ}$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

Answer: 89

Solution:



Note that $\triangle AOC$ is isosceles, with $AO = CO = 10$ and $AC = 12$. Then draw the altitude of $\triangle AOC$ from $\overline{AC}$, and using the Law of Sines, deduce that AP, which is the circumradius of $\triangle AOC$, is $\frac{25}{4}$. To find AQ, observe that since $\triangle AOP$ and $\triangle AOC$ are isosceles (two sides are circumradii), and $\angle PAO \cong \angle POA \cong \angle POC \cong \angle ABC$ since $\angle AOC$ is an exterior angle to isosceles triangle $\triangle BOC$. Then since $\angle ABC \cong \angle BAQ$, and $\triangle ABC$ and $\triangle BAQ$ are both right (they are both inscribed in a semicircle), they're congruent, so $AQ = BC = \sqrt{20^2 - 12^2} = 16$ by the Pythagorean Theorem. Then $\frac{AP}{AQ} = \frac{\frac{25}{4}}{16} = \frac{25}{64}$, and our answer is $\boxed{89}$.

8. Let triangle $\triangle ABC$ have $AB = 17$, $BC = 14$, $CA = 12$. Let M_A, M_B, M_C be midpoints of $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ respectively. Let the angle bisectors of A, B, and C intersect $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ at P, Q, and R, respectively. Reflect M_A about $\overline{AP}$, M_B about $\overline{BQ}$, and M_C about $\overline{CR}$ to obtain M'_A, M'_B, M'_C, respectively. The lines $\overleftrightarrow{AM'_A}$, $\overleftrightarrow{BM'_B}$, and $\overleftrightarrow{CM'_C}$ will then intersect $\overline{BC}, \overline{AC}$, and $\overline{AB}$ at D, E, and F, respectively. Given that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ concur at a point K inside the triangle, in simplest form, the ratio $[KAB] : [KBC] : [KCA]$ can be written in the form $p : q : r$, where p, q and r are relatively prime positive integers and $[XYZ]$ denotes the area of $\triangle XYZ$. Compute $p + q + r$.

Answer: 629

Solution: (On the version of the test sent out to contestants, the last line initially said "Compute $p^2 + q^2 + r^2$." We apologize for the error!)
First, notice that the effect of reflecting medians over angle bisectors is that angles are preserved – in particular, $\angle M_A AP = \angle PAD$ so because AP is an angle bisector, $\angle M_A AB = \angle DAC$, etc for all 3 sides. Also notice that because AP is a median:

$$1 = \frac{BM_A}{M_A C} = \frac{[BAM_A]}{[CAM_A]} = \frac{\frac{1}{2}(AB)(AM_A)\sin\angle BAM_A}{\frac{1}{2}(AC)(AM_A)\sin\angle CAM_A} = \frac{AB\sin\angle BAM_A}{AC\sin\angle CAM_A}$$

so $AB\sin\angle BAM_A = AC\sin\angle CAM_A$. Then using the $\frac{1}{2}ab\sin C$ formula for area of a triangle:

$$\frac{BD}{DC} = \frac{[ABD]}{[ADC]} = \frac{AB\sin\angle BAD}{AC\sin\angle DAC} = \frac{AB\sin\angle CAM_A}{AC\sin\angle BAM_A} = \left(\frac{AB}{AC}\right)^2 \frac{AB\sin\angle BAM_A}{AC\sin\angle CAM_A} = \left(\frac{AB}{AC}\right)^2$$

so the analogous ratios for the other sides of the triangle are

$$\frac{BD}{DC} = \left(\frac{AB}{AC}\right)^2$$
$$\frac{CE}{EA} = \left(\frac{BC}{AB}\right)^2$$
$$\frac{AF}{FB} = \left(\frac{AC}{BC}\right)^2.$$

To find the area ratios, we can assign mass points: $A : BC^2, B : AC^2, C : AB^2$. Then the masses of the other points are:

$$D : AC^2 + AB^2, E : AB^2 + BC^2, F : AC^2 + BC^2 \implies K : AB^2 + BC^2 + AC^2$$

then, if we let $m(K)$ denote the mass of K:

$$[KAB] : [KBC] : [KCA] = \frac{KF}{CF} : \frac{KD}{AD} : \frac{KE}{BE} = \frac{AB^2}{m(K)} : \frac{BC^2}{m(K)} : \frac{AC^2}{m(K)} = 17^2 : 14^2 : 12^2$$

so our final answer is $17^2 + 14^2 + 12^2 = \boxed{629}$.

9. The *Fibonacci numbers* F_n are defined as $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$. Let A be the minimum area of a (possibly degenerate) convex polygon with 2020 sides, whose side lengths are the first 2020 Fibonacci numbers $F_1, F_2, \dots, F_{2020}$ (in any order). A *degenerate convex polygon* is a polygon where all angles are $\leq 180°$. If A can be expressed in the form $\frac{\sqrt{(F_a-b)^2-c}}{d}$, where a, b, c and d are positive integers, compute the minimal possible value of $a + b + c + d$.

Answer: 2029

Solution: *Lemma 1:* Any shape that is not a triangle has nonminimal area.
Proof: Assume that a non-triangle shape has minimal area. Then there exist four corners (with no 3 corners collinear) that can be viewed as forming the vertices of a quadrilateral. Call this quadrilateral $ABCD$, and let $a = AB$, $b = BC$, $c = CD$, $d = DA$. Without loss of generality, let $\angle ABC + \angle CDA \geq 180°$. By Bretschneider's Formula, the area of $ABCD$ is $\sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2(\frac{\angle B+\angle D}{2})}$, where s is semiperimeter. Let A', B', C', D' be positions of A, B, C, D such that a, b, c, d aren't changed, and $\angle B + \angle D$ is maximized, while ensuring that all angles are $\leq 180°$. Note that since $ABCD$ is a non-triangle quadrilateral, it is always possible to find A', B', C', D' such that $\angle B' + \angle D' > \angle B + \angle D$. By Bretschneider's Formula, since $\frac{\angle B+\angle D}{2}$ lies in Quadrant II, $\cos(\frac{\angle B+\angle D}{2})$ becomes more negative so the $abcd\cos^2(\frac{\angle B+\angle D}{2})$ term increases, meaning the resulting quadrilateral is smaller in area. We then have an overall reduction in area without changing the side lengths; we thus have an n-gon of smaller area, contradicting the assumption that this shape is minimal, thus any non-triangle is minimal.

Lemma 2: Let $\triangle ABC$ have a constant perimeter $2p + 1$ and integer side lengths. Then the triangle of minimal area has side lengths $1, p, p$.
Proof: Assume that this minimal triangle has side lengths a, b, c, with $1 < a \leq b \leq c$. By Heron's Formula, we have that $A = \sqrt{s(s-a)(s-b)(s-c)}$, where s denotes semiperimeter. Note that $a' = a - 1, b' = b + 1$ yields a triangle of area $\sqrt{s(s-a+1)(s-b-1)(s-c)} = \sqrt{s(s^2 - bs - s - as + ab + a + s - b - 1)(s-c)} = \sqrt{s((s-a)(s-b) + a - b - 1)(s-c)}$ (This does not violate the Triangle Inequality if the original triangle does not violate Triangle Inequality). Since $a \leq b$, we have that $a - b - 1 \leq -1 < 0$, so this triangle with side lengths $a-1, b+1, c$ has a smaller area. This contradicts our assumption that the minimal triangle has side lengths a, b, c. Then we must have $a = 1$. By Triangle Inequality, we must have $b = c = p$. Thus, the minimal area triangle has side lengths $1, p, p$.

By Lemmas 2 and 1, if a triangle with side lengths $1, p, p$ can be constructed, it must be the one with minimal area (since such a triangle would have odd perimeter). Note that because $F_{2018} + F_{2019} = F_{2020}$, $F_{2015} + F_{2016} = F_{2017}, \dots, F_{3n-1} + F_{3n} = F_{3n+1}, \dots, F_2 + F_3 = F_4$, we can thus construct a triangle with one side as the sum of all terms of the form F_{3n+1} $(n > 0)$, another side as the sum of all terms of the form F_{3n}, F_{3n-1} $(n > 0)$, and the last as $F_1 = 1$ to generate a triangle of side lengths $1, p, p$. We can calculate its area (noting that $1 + p + p = F_{2022} - 1$) as two right triangles of base $\frac{1}{2}$ and hypotenuse $\frac{F_{2022}-2}{2}$. By the Pythagorean Theorem, this has height $\sqrt{\frac{(F_{2022}-2)^2}{4} - \frac{1}{4}} = \frac{\sqrt{(F_{2022}-2)^2-1}}{2}$, so our triangle's area is $\frac{\sqrt{(F_{2022}-2)^2-1}}{4}$, and the answer is $2022 + 2 + 1 + 4 = \boxed{2029}$.

10. Let E be an ellipse where the length of the major axis is 26, the length of the minor axis is 24, and the foci are at points R and S. Let A and B be points on the ellipse such that $RASB$ forms a non-degenerate quadrilateral, $\overleftrightarrow{RA}$ and $\overleftrightarrow{SB}$ intersect at P with segment $\overline{PR}$ containing A, and

$\overleftrightarrow{RB}$ and $\overleftrightarrow{AS}$ intersect at Q with segment $\overline{QR}$ containing B. Given that $RA = AS$, $AP = 26$, the perimeter of the non-degenerate quadrilateral $RPSQ$ is $m + \sqrt{n}$, where m and n are integers. Compute $m + n$.

Answer: 5362

Solution: We observe the following known as Urquhart's Theorem. We will provide an elementary proof of the fact and refer the reader to an enlightening, albeit non-elementary proof:
Claim: P and Q lie on an ellipse with foci R and S.
Proof 1. We wish to show that $RP + PS = RQ + QS$. One equivalent formulation of this is to show that there exists a circle externally tangent to $\overline{RP}, \overline{SP}, \overline{SQ}, \overline{RQ}$. Then by Pitot's theorem this will follow. Let Γ_1 be the R-excircle of $\triangle RPB$ and Γ_2 be the R-excircle of $\triangle RQA$. We show that Γ_1 and Γ_2 coincide. Choose S_1 on $\overline{RP}$ such that $AS_1 = AS$, and S_2 on $\overline{RQ}$ such that $BS_2 = BS$. Then since A and B lie on the ellipse whose foci are R and S, it follows that $RS_1 = RS_2$ so $\triangle RS_1S_2$ is isosceles. The angle bisector of $\angle PQS$ passes through the center of circle Γ_2 and the angle bisector of $\angle CDE$ passes through the center of circle Γ_1. The angle bisector of $\angle PRQ$ passes through the centers of both circles. But since $\triangle AS_1S$, $\triangle BSS_2$, and $\triangle RS_1S_2$ are all isosceles, the angle bisectors correspond to perpendicular bisectors of $\overline{SS_1}$, $\overline{SS_2}$, and $\overline{S_1S_2}$, respectively. This implies that all three lines are concurrent so the centers of the three circles coincide. Both circles must be tangent to lines $\overleftrightarrow{PR}$ and $\overleftrightarrow{QR}$, and there's only one such circle, so the circle is unique.
Proof 2. See `https://www.tandfonline.com/doi/pdf/10.1080/00029890.2007.11920482`. It may also be helpful to learn the Liouville-Arnold Theorem.
First, we compute that the distance between the foci is $\sqrt{26^2 - 24^2} = 10$ and we can compute that $RA = AS = 13$. Let C be the center of the ellipse. Let $\angle RAC = \theta$. We have $\cos(\angle PAS) = \cos(180 - 2\theta) = -\cos(2\theta) = \sin^2(\theta) - \cos^2(\theta) = -\frac{119}{169}$. Using the Law of Cosines on $\triangle RPS$, we can compute that $PS = \sqrt{13^2 + 26^2 + 4 \cdot 119} = \sqrt{1321}$, so the perimeter is $2(13 + \sqrt{1321} + 26) = 78 + \sqrt{5284}$, and our answer is $\boxed{5362}$.