Time limit: 60 minutes.
Instructions: This test contains 10 short answer questions. All answers are positive integers. Only submitted answers will be considered for grading.
No calculators.

1. Marisela is putting on a juggling show! She starts with 1 ball, tossing it once per second. Lawrence tosses her another ball every five seconds, and she always tosses each ball that she has once per second. (Marisela tosses her first ball at the 1st second, and starts tossing the second ball at the 6th second. Tosses at the 60th second also count.) Compute the total number of tosses Marisela has made one minute after she starts juggling.

2. Let a and b be the roots of the polynomial $x^2 + 2020x + c$. Given that $\frac{a}{b} + \frac{b}{a} = 98$, compute $\sqrt{c}$.

3. The graph of the degree 2021 polynomial $P(x)$, which has real coefficients and leading coefficient 1, meets the x-axis at the points $(1, 0), (2, 0), (3, 0), \ldots, (2020, 0)$ and nowhere else. The mean of all possible values of $P(2021)$ can be written in the form $a!/b$, where a and b are positive integers and a is as small as possible. Compute $a + b$.

4. Let φ be the positive solution to the equation

$$x^2 = x + 1.$$

For $n \geq 0$, let a_n be the unique integer such that $\varphi^n - a_n\varphi$ is also an integer. Compute

$$\sum_{n=0}^{10} a_n.$$

5. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that for all $x, y \in \mathbb{R}^+$, $f(x)f(y) = f(xy) + f\left(\frac{x}{y}\right)$, where $\mathbb{R}^+$ represents the positive real numbers. Given that $f(2) = 3$, compute the last two digits of $f\left(2^{2^{2020}}\right)$.

6. Given that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, the value of

$$\sum_{n=3}^{10} \frac{\binom{n}{2}}{\binom{n}{3}\binom{n+1}{3}}$$

can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

7. Let a, b, and c be real numbers such that $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ and $abc = 5$. The value of

$$\left(a - \frac{1}{b}\right)^3 + \left(b - \frac{1}{c}\right)^3 + \left(c - \frac{1}{a}\right)^3$$

can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

8. Compute the smallest real value C such that the inequality

$$x^2(1+y)+y^2(1+x) \leq \sqrt{(x^4+4)(y^4+4)}+C$$

holds for all real x and y.

9. There is a unique unordered triple (a,b,c) of two-digit positive integers a, b, and c that satisfy the equation

$$a^3+3b^3+9c^3=9abc+1.$$

Compute $a+b+c$.

10. For $k \geq 1$, define $a_k = 2^k$. Let

$$S=\sum_{k=1}^{\infty} \cos^{-1}\left(\frac{2a_k^2-6a_k+5}{\sqrt{(a_k^2-4a_k+5)(4a_k^2-8a_k+5)}}\right).$$

Compute $\lfloor 100S \rfloor$.