Time limit: 60 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written inside the boxes on the answer sheet will be considered for grading.
No calculators.

1. A fair coin is repeatedly flipped until 2019 consecutive coin flips are the same. Compute the probability that the first and last flips of the coin come up differently.

2. Sylvia has a bag of 10 coins. Nine are fair coins, but the tenth has tails on both sides. Sylvia draws a coin at random from the bag and flips it without looking. If the coin comes up tails, what is the probability that the coin she drew was the 2-tailed coin?

3. There are 15 people at a party; each person has 10 friends. To greet each other each person hugs all their friends. How many hugs are exchanged at this party?

4. There exists one pair of positive integers a, b such that $100 > a > b > 0$ and $\frac{1}{a} + \frac{1}{b} = \frac{2}{35}$. Find $a + b$.

5. Let $2^{1110} \equiv n \pmod{1111}$ with $0 \leq n < 1111$. Compute n.

6. Define $f(n) = \frac{n^2+n}{2}$. Compute the number of positive integers n such that $f(n) \leq 1000$ and $f(n)$ is the product of two prime numbers.

7. Call the number of times that the digits of a number change from increasing to decreasing, or vice versa, from the left to right while ignoring consecutive digits that are equal the *flux* of the number. For example, the flux of 123 is 0 (since the digits are always increasing from left to right) and the flux of 12333332 is 1, while the flux of 9182736450 is 8. What is the average value of the flux of the positive integers from 1 to 999, inclusive?

8. For a positive integer n, define $\phi(n)$ as the number of positive integers less than or equal to n that are relatively prime to n. Find the sum of all positive integers n such that $\phi(n) = 20$.

9. Let $z = \frac{1}{2}\left(\sqrt{2} + \mathrm{i}\sqrt{2}\right)$. The sum
$$\sum_{k=0}^{13} \frac{1}{1 - z\mathrm{e}^{k\cdot \mathrm{i}\pi/7}}$$
can be written in the form $a - b\mathrm{i}$. Find $a + b$.

10. Let $S(n)$ be the sum of the squares of the positive integers less than and coprime to n. For example, $S(5) = 1^2 + 2^2 + 3^2 + 4^2$, but $S(4) = 1^2 + 3^2$.

    Let $p = 2^7 - 1 = 127$ and $q = 2^5 - 1 = 31$ be primes. The quantity $S(pq)$ can be written in the form
$$\frac{p^2q^2}{6}\left(a - \frac{b}{c}\right)$$
    where a, b, and c are positive integers, with b and c coprime and $b < c$. Find a.