Power Round

Welcome to the power round! This year's topic is the theory of orthogonal polynomials.

I. You should order your papers with the answer sheet on top, and you should number papers addressing the same question. Include your **Team ID** at the top of each page you submit.

II. You may reference anything stated or cited earlier in the test, even if you do not understand it. You may not reference outside sources or proofs to answers not given on the same page.

III. You have **60 minutes** to answer 16 questions, cumulatively worth **100 points**. Good luck!

0 Introduction

As far as I am concerned, the primary purpose of a power round is to show you, the students, what it is like to do mathematics at the university level. I take my responsibility as a representative of higher education very seriously, and I hope you will enjoy some "Eureka" moments during this test. I encourage you most of all to read everything, for even unsolved problems may be understood later.

I have included a series of conceptual questions on this test that account for approximately one quarter of all possible points. You should try to figure out the motivation behind the question and review the recent material to determine how best to answer. If a problem does not explicitly require demonstration of a proof or computation, you may optionally choose to supplement your answer with either. However, if you see terms such as *prove*, *verify*, or *determine*, proof techniques are required for full points. Problems with spots provided on the answer sheet require no explanation.

Think critically. A mathematician always knows exactly what she is talking about, and you may try to do the same by paying careful attention to the definitions. A good beginning is your best way to partial credit on the harder questions, so make sure you do know what you are talking about.

The set of **natural numbers** is $\{0, 1, 2, \dots\}$. A natural (number) n is an element of this set.
The set of **real numbers** is $\mathbb{R}$. A real number c is a number found on the number line.

A **polynomial** is a function p, from the real numbers to the real numbers, that has the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ for real numbers $a_n, a_{n-1}, \dots, a_0$ (where $a_n \neq 0$ or $n = 0$) and natural n. The **degree** of the polynomial is n. A **root** of the polynomial is a value c such that $p(c) = 0$. The **Fundamental Theorem of Algebra** states that polynomials of degree n have at most n roots. By convention, this test only contains polynomials with x as the parameter.

One term I use but do not define is *function space*. Formally, I mean a vector space (over the real numbers) whose elements are functions sharing domain and codomain. Informally, I mean a set of real-valued functions you can add together. Some examples of function spaces are the set of real numbers $\mathbb{R}$ (i.e., the collection of constant functions from $\mathbb{R}$ to $\mathbb{R}$), the set of polynomials $\mathcal{P}$, and the set of functions f_a^b of the form $f_a^b(x) = a \cdot e^x + b \cdot \sqrt[3]{x}$ for some real numbers a and b.

1 Functionals (24 pts)

All areas of mathematics have some concept of *mapping* crucial to development of the theory. A **function** maps elements from one value to another. The **domain** is the set of values on which the function is defined, and the **codomain** is the set of possible values into which the function maps. The **range** is the precise set of values the function can achieve. The function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^2 + \sqrt{x} + 1$ has domain $[0, \infty)$, codomain $\mathbb{R}$, and range $[1, \infty)$. Henceforth, we will speak only of real-valued functions— functions whose codomain is the set of real numbers.

- A **functional** maps (real-valued) functions to real numbers. Its domain is a function space, and its codomain is the set of real numbers.
- A **linear** functional is a functional $\mathbf{T}$ satisfying:
  1. For all functions f and g in its domain, $\mathbf{T}(f + g) = \mathbf{T}(f) + \mathbf{T}(g)$.
  2. For all real numbers c and functions f in its domain, $c \cdot \mathbf{T}(f) = \mathbf{T}(c \cdot f)$.

(The term function space is defined in the introduction.)

For instance, a functional $\mathbf{Lead} : \mathcal{P} \to \mathbb{R}$ may take a polynomial as input and return the coefficient of the term of highest degree, mapping $(2x^2 + 3x + 1) \mapsto 2$ and $(2x^2 + 3x^3 + 1) \mapsto 3$.

1. The following functionals are linear. Use the properties of linearity to determine the answers.
   (a) **[1]** If $\mathbf{A}(1) = 2$ and $\mathbf{A}(x) = 3$, find $\mathbf{A}(2x - 1)$.
   (b) **[1]** If $\mathbf{B}(2x + 1) = 0$, $\mathbf{B}(2x^2 - 4x + 6) = 6$, and $\mathbf{B}(x^3 + 7x) = 8$, find $\mathbf{B}(x^3 + x^2 + x + 1)$.
   (c) **[1]** If $\mathbf{C}(\cos x) = 4$ and $\mathbf{C}(\sin x) = 2\sqrt{3}$, find $\mathbf{C}(\sin(x + \frac{\pi}{3}))$.
   (d) **[1]** If $\mathbf{D}(x^k) = 2k - 1$ for all natural k, find $\mathbf{D}((2x - 1)^{10})$.

Solution to Problem 1:

1. (a) $\mathbf{A}(2x - 1) = 2\mathbf{A}(x) - \mathbf{A}(1) = \boxed{4.}$
   (b) $\mathbf{B}(x^3 + x^2 + x + 1) = \mathbf{B}(x^3 + 7x) + \frac{1}{2}\mathbf{B}(2x^2 - 4x + 6) - 2\mathbf{B}(2x + 1) = \boxed{11.}$
   (c) $\mathbf{C}(\sin(x + \frac{\pi}{3})) = \frac{1}{2}\mathbf{C}(\sin x) + \frac{\sqrt{3}}{2}\mathbf{C}(\cos x) = \boxed{3\sqrt{3}.}$
   (d) Note $\mathbf{D} = 2\mathbf{Ev}_1 \circ \frac{d}{dx} - \mathbf{Ev}_1$ for polynomials. Thus,

$$\begin{aligned}\mathbf{D}((2x-1)^{10}) &= 2\mathbf{Ev}_1(\tfrac{d}{dx}((2x-1)^{10})) - \mathbf{Ev}_1((2x-1)^{10}) \\ &= 2\mathbf{Ev}_1(20(2x-1)^9) - \mathbf{Ev}_1((2x-1)^{10}) = \boxed{39.}\end{aligned}$$

2. For each of the following, prove the statement or provide a counterexample.
   (a) **[2]** Every (real-valued) function is a functional.
   (b) **[2]** Every functional is a (real-valued) function.
   (c) **[2]** The codomain of a functional is always contained in its domain.

Solution to Problem 2:

2. (a) Not all functions are functionals. For instance, a function f : Animals $\to \mathbb{R}$ mapping an animal to the number of legs it has is not a functional, since animals are not functions.

   (b) Every functional is a function, since a functional is just a function that takes a function as an input.

   (c) The codomain of a functional is not necessarily contained in its domain. For instance, the functional **Lead** restricted to nonconstant polynomials does not contain the real numbers in its domain.

3. Answer each of the following questions in a clear and concise manner.

   (a) [**3**] Why is it conventional to use codomain instead of range when defining a function?

   (b) [**3**] Is there such a thing as an "inverse functional"? For instance, can you construct a mapping $\Lambda : \mathbb{R} \to \mathcal{P}$ that is an inverse to **Lead**? What limitations, if any, are there?

   (c) [**4**] Consider the functional $\mathbf{Ev}_2(\varphi) = \varphi(2)$ and the function $f(x) = xy+z$, where y and z are fixed constants. Explain the difference in meaning between $\mathbf{Ev}_2(f)$ and $\mathbf{Ev}_2(xy+z)$. Now, consider the function $g(x) = x^2 + x$. Explain the difference in meaning between $\mathbf{Ev}_2(g)$ and $\mathbf{Ev}_2(x^2 + x)$. Is either mistake acceptable? Is it possible to avoid this type of mistake without separately defining a function, as done here?

Solution to Problem 3:

3. (a) Here are three possible reasons, roughly ordered to the writer's preference.

      i. If instead it were conventional to use range when defining a function, one would need to know a priori the set of possible values the function could achieve. Since the function is only just being defined, it seems reasonable to presume that those values may not all be known. Certainly in some cases (extrema of bounded continuous functions), the range provides much information about the function itself.

      ii. Without codomain, there would be no concept of surjectivity. Isomorphism theorems would weaken, and much of algebra would collapse into trivialities. There's a lot to say about a space even when function mappings into and out of the space don't line up exactly (e.g., homology).

      iii. It's easier to define the function without thinking about what exactly the function is— only what it could be.

   (b) There is not exactly an inverse functional, but there is something partial. The mapping $\Lambda : c \mapsto c \cdot x^2$, for instance, acts as a right inverse to **Lead**; that is, $\mathbf{Lead}(\Lambda(c)) = c$ for all real numbers c. It is impossible to create a left inverse to **Lead** because it is a noninjective function from $\mathcal{P}$ to $\mathbb{R}$; that is, for any given nonzero real number, there are multiple polynomials yielding it upon application of **Lead**. A right inverse does exist at 0 (and only 0), though.

   (c) $\mathbf{Ev}_2(f)$ means the result of applying the functional $\mathbf{Ev}_2$ to the function f, which is well-defined to be $f(2) = 2y + z$. $\mathbf{Ev}_2(xy + z)$ literally means the result of applying the functional $\mathbf{Ev}_2$ to the function $xy+z$; however, $xy+z$ is an expression and not a function, so the meaning is not well-defined here. Only with the understanding that $xy + z$ is shorthand for the function $f(x) = xy + z$ may any meaning be ascribed to the expression, and in that case, the result is, again, $2y + z$. Similarly, $\mathbf{Ev}_2(g)$ and

$\mathbf{Ev}_2(x^2+x)$ both evaluate to $g(2) = 6$, but the second expression is unclear without context upon realizing x could be a function. The mistake is more acceptable in the latter case, for it can be expected that x is the active variable in the expression; however, ambiguity arises in the former case where, for instance, $h(z) = xy + z$ would yield a different output than f when passed through $\mathbf{Ev}_2$. One may write $\mathbf{Ev}_2(x \mapsto (xy+z))$ to resolve this ambiguity without the need to define a function.

We will now restrict our attention to $\mathcal{P}$, the function space of polynomial functions. For any real numbers $a < b$, we define a linear functional $\mathbf{Int}_{[a,\,b]}$ with domain $\mathcal{P}$ as follows.

$$\text{For any natural } n,\ \mathbf{Int}_{[a,\,b]}(x^n) = \frac{b^{n+1} - a^{n+1}}{n+1}.$$

Using the definition of linearity, we may extend this definition to all polynomials. For instance,

$$\begin{aligned}\mathbf{Int}_{[0,\,1]}(6x^2 + 2x + 7) &= 6 \cdot \mathbf{Int}_{[0,\,1]}(x^2) + 2 \cdot \mathbf{Int}_{[0,\,1]}(x) + 7 \cdot \mathbf{Int}_{[0,\,1]}(1) \\ &= 6 \cdot \left(\frac{1^3 - 0^3}{3}\right) + 2 \cdot \left(\frac{1^2 - 0^2}{2}\right) + 7 \cdot \left(\frac{1^1 - 0^1}{1}\right) \\ &= 2 + 1 + 7 = 10.\end{aligned}$$

In fact, the definition for $\mathbf{Int}_{[a,\,b]}$ can be extended to functions beyond just polynomials. A purely formulaic reason for this is that *well-behaved* functions can be approximated *very well* by polynomial functions. But for this test, it is only necessary to know how to apply the functional to polynomials.

4. (a) **[1]** Evaluate $\mathbf{Int}_{[0,\,2]}(3x^2 + 2x)$.
   (b) **[1]** Evaluate $\mathbf{Int}_{[1,\,3]}(x^7 + x^3)$.
   (c) **[2]** Prove that for any polynomial p and real numbers $a < b < c$,

$$\mathbf{Int}_{[a,\,b]}(p) + \mathbf{Int}_{[b,\,c]}(p) = \mathbf{Int}_{[a,\,c]}(p).$$

Solution to Problem 4:

4. (a) The expression is evaluated as follows.

$$\begin{aligned}\mathbf{Int}_{[0,\,2]}(3x^2 + 2x) &= 3 \cdot \mathbf{Int}_{[0,\,2]}(x^2) + 2 \cdot \mathbf{Int}_{[0,\,2]}(x) \\ &= 3 \cdot \left(\frac{2^3 - 0^3}{3}\right) + 2 \cdot \left(\frac{2^2 - 0^2}{2}\right) \\ &= 8 + 4 = \boxed{12.}\end{aligned}$$

   (b) The expression is evaluated as follows.

$$\begin{aligned}\mathbf{Int}_{[1,\,3]}(x^7 + x^3) &= \mathbf{Int}_{[1,\,3]}(x^7) + \mathbf{Int}_{[1,\,3]}(x^3) \\ &= \frac{3^8 - 1^8}{8} + \frac{3^4 - 1^4}{4} \\ &= 820 + 20 = \boxed{840.}\end{aligned}$$

(c) For any natural n,

$$\begin{aligned}\mathrm{Int}_{[a,\,b]}(x^n) + \mathrm{Int}_{[b,\,c]}(x^n) &= \frac{b^{n+1} - a^{n+1}}{n+1} + \frac{c^{n+1} - b^{n+1}}{n+1} \\ &= \frac{c^{n+1} - a^{n+1}}{n+1} \\ &= \mathbf{Int}_{[a,\,c]}(x^n).\end{aligned}$$

It then follows from the linearity of $\mathbf{Int}_{[a,\,b]}$, $\mathbf{Int}_{[b,\,c]}$, and $\mathbf{Int}_{[a,\,c]}$ that the equivalence holds for any polynomial p.

It may be useful later to note that for any real numbers $a < b$ and polynomial p, there exists a value c satisfying $a < c < b$ such that $\mathbf{Int}_{[a,\,b]}(p) = (b - a) \cdot f(c)$. This is the **Mean Value Theorem**.

2 Simple Orthogonality (25 pts)

Two polynomials f and g are ***simply* orthogonal** if

$$\mathbf{Int}_{[-1,1]}(f \cdot g) = 0.$$

A set of polynomials is simply orthogonal if any distinct two of its elements are simply orthogonal.

5. Which of the following sets are simply orthogonal?
   (a) **[2]** $\{1, x\}$; $\{1, x^2\}$; $\{x, x^2\}$; $\{1, x^2 - \frac{1}{3}\}$; $\{x^2 - 1, x^3\}$
   (b) **[2]** $\{1, x, x^2\}$; $\{1, x, \frac{3}{2}x^2 - \frac{1}{2}\}$; $\{1, x^2 - 1, x^3\}$; $\{1, x^2 - \frac{1}{3}, x^3 + 2x\}$

Solution to Problem 5:

5. (a) The simply orthogonal sets are $\{1, x\}$, $\{x, x^2\}$, $\{1, x^2 - \frac{1}{3}\}$, and $\{x^2 - 1, x^3\}$.
   (b) The simply orthogonal sets are $\{1, x, \frac{3}{2}x^2 - \frac{1}{2}\}$ and $\{1, x^2 - \frac{1}{3}, x^3 + 2x\}$.

6. For each of the following, find a nonzero function satisfying the given condition, or prove none exist.
   (a) **[2]** Find a linear polynomial simply orthogonal to each of 1, $x^2 - \frac{1}{3}$, and $x^3 + 2x$.
   (b) **[2]** Find a cubic polynomial simply orthogonal to each of 1, x, and $x^2 - \frac{1}{3}$.

Solution to Problem 6:

6. (a) There is no such polynomial. Suppose one does exist and is $ax + b$, for real a and b. Then,

$$\mathbf{Int}_{[-1,\,1]}(ax + b) = 0 \implies a \cdot \left(\frac{1^2 - (-1)^2}{2}\right) + b \cdot (1 - (-1)) = 0 \implies b = 0, \text{ and}$$
$$\mathbf{Int}_{[-1,\,1]}(ax^4 + bx^3 + 2ax^2 + 2bx) = 0 \implies a \cdot (\tfrac{2}{5} + 2 \cdot \tfrac{2}{3}) + b \cdot (0 + 2 \cdot 0) = 0 \implies a = 0.$$

   Thus, the "linear" polynomial is in fact the zero polynomial, and by way of contradiction, we have proved our claim.

   (b) Any nonzero multiple of the polynomial $\boxed{x^3 - \frac{3}{5}x}$ would suffice.

A function f is **symmetric** if $f(c) = f(-c)$ for all c in its domain. A function f is **antisymmetric** if $f(c) = -f(-c)$ for all c in its domain. For polynomials, the domain is all real numbers.

7. Answer the following questions on symmetric and antisymmetric polynomials.
   (a) **[2]** Prove that if a polynomial is symmetric, then each of its terms has even degree.
   (b) **[2]** Prove that if a polynomial is antisymmetric, then each of its terms has odd degree.
   (c) **[1]** Prove, for any antisymmetric polynomial p, that $\mathbf{Int}_{[-1,\,1]}(p) = 0$.

Solution to Problem 7:

7. (a) Let p be a symmetric polynomial. Then, $p(c) - p(-c) = 0$ for all real c, implying the polynomial $p(x) - p(-x)$ is in fact the zero function (the Fundamental Theorem of Algebra implies a polynomial may have only finitely many roots). But suppose there is a term of odd degree in p; then $p(x) - p(-x)$ has the same term of odd degree (with coefficient doubled), a contradiction. Therefore, p contains only terms

of even degree.

(b) Let p be an antisymmetric polynomial. Then, $p(c)+p(-c)=0$ for all real c, implying the polynomial $p(x)+p(-x)$ is in fact the zero function (the Fundamental Theorem of Algebra implies a polynomial may have only finitely many roots). But suppose there is a term of even degree in p; then $p(x)+p(-x)$ has the same term of even degree (with coefficient doubled), a contradiction. Therefore, p contains only terms of odd degree.

(c) Note first that for any odd k, $k+1$ is even and

$$\mathbf{Int}_{[-1,\,1]}(x^k) = \frac{1^{k+1}-(-1)^{k+1}}{k+1} = 0.$$

Let p be an antisymmetric polynomial. Then p only has terms of odd degree. It follows from linearity of $\mathbf{Int}_{[-1,\,1]}$ that $\mathbf{Int}_{[-1,\,1]}(p)=0$.

The **Legendre polynomials** comprise a sequence of simply orthogonal polynomials, the n^{th} of which is degree n. Any one Legendre polynomial is orthogonal to any other Legendre polynomial. They begin $P_0(x)=1$ and $P_1(x)=x$ and are subject to the standardization $P_n(1)=1$ for all natural n. They are uniquely determined by this definition, but an equivalent definition is

$$P_{n+1}(x) = \left(\frac{2n+1}{n+1}\right) xP_n(x) - \left(\frac{n}{n+1}\right) P_{n-1}(x) \text{ for all natural } n>0.$$

8. (a) **[2]** Compute $P_2(x)$, $P_3(x)$ and $P_4(x)$.
   (b) **[2]** Prove that all terms of a Legendre polynomial have the same parity of degree.
   (c) **[2]** Verify with computation or proof that P_4 is orthogonal to P_0, P_1, P_2, and P_3.

Solution to Problem 8:

8. (a) $P_2(x)=\frac{3}{2}x^2-\frac{1}{2}$, $P_3(x)=\frac{5}{2}x^3-\frac{3}{2}x$, and $P_4(x)=\frac{35}{8}x^4-\frac{15}{4}x^2+\frac{3}{8}$.

(b) Let n be a natural number. We claim that all terms of the n^{th} Legendre polynomial have the same parity as n. This is true for the zeroth and first Legendre polynomials. Suppose it is true for Legendre polynomials k and $k-1$, where $k>0$. Then, from the recursive definition of Legendre polynomials, P_{k+1} has terms of the same parity as P_{k-1} and terms of the opposite parity as P_k. If $k+1$ is odd, this implies P_{k+1} only has terms of odd parity, and if $k+1$ is even, this implies P_{k+1} only has terms of even parity. By the principle of mathematical induction, our claim holds, and we conclude that the problem statement is true.

(c) By a property of antisymmetry, it is immediate that P_4 is orthogonal to P_1 and P_3. Note, then, that P_4 is orthogonal to P_0:

$$\mathbf{Int}_{[-1,\,1]}(\tfrac{35}{8}x^4-\tfrac{15}{4}x^2+\tfrac{3}{8}) = \tfrac{35}{8}\cdot\tfrac{2}{5}-\tfrac{15}{4}\cdot\tfrac{2}{3}+\tfrac{3}{8}\cdot 2 = \tfrac{7}{4}-\tfrac{5}{2}+\tfrac{3}{4}=0.$$

And note that P_4 is orthogonal to P_2:

$$\mathbf{Int}_{[-1,\,1]}(\tfrac{105}{16}x^6-\tfrac{125}{16}x^4+\tfrac{39}{16}x^2-\tfrac{3}{16}) = \tfrac{105}{16}\cdot\tfrac{2}{7}-\tfrac{125}{16}\cdot\tfrac{2}{5}+\tfrac{39}{16}\cdot\tfrac{2}{3}-\tfrac{3}{16}\cdot 2=0.$$

9. (a) **[2]** Express x^3 as a sum of distinct nonzero multiples of Legendre polynomials.

   (b) **[4]** The team across the room got a different answer for part (a). Prove them wrong.

Solution to Problem 9:

9. (a) $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$.

   (b) Suppose the team is right and has representation $Sum(x) = x^3$, where $Sum(x)$ is a sum of distinct nonzero multiples of Legendre polynomials. Then, $Sum(x) - \frac{2}{5}P_3(x) - \frac{3}{5}P_1(x) = x^3 - x^3 = 0$ is a sum of distinct nonzero multiples of Legendre polynomials after combining like terms, and $Sum(x) - \frac{2}{5}P_3(x) - \frac{3}{5}P_1(x)$ is nonzero because the team got a different answer. Therefore, there must be a greatest n such that P_n is present in the sum (with nonzero coefficient). But then $0 = c \cdot x^n +$ lower order terms for some nonzero real number c. As we have reached a contradiction, we must conclude the other team is wrong.

For more on Legendre polynomials, go directly to Section 4. For more on orthogonal polynomials in general, continue to Section 3.

3 Orthogonal Polynomials (28 pts)

More generally, orthogonal polynomials arise in a space equipped with an **inner product**. Inner products are two-variable functions that are linear functionals in either variable. In particular, we are concerned with inner products of the following form, where a and b are real numbers and w is a polynomial that is positive throughout the interval (a, b).

$$\langle p, q\rangle = \mathbf{Int}_{[a,\,b]}(p\cdot q\cdot w) \text{ for all polynomials } p \text{ and } q.$$

Two polynomials p and q are **orthogonal** if $\langle p, q\rangle = 0$. As with Legendre polynomials, we may construct sequences of orthogonal polynomials. Henceforth, the term **orthogonal polynomials** refers to a sequence of polynomials orthogonal to each other, the n^{th} of which is degree n. The **Gram-Schmidt process** provides an immediate method of creating some orthogonal polynomials:

$$p_n(x) = x^n - \sum_{k=0}^{n-1} \frac{\langle x^n, p_k\rangle}{\langle p_k, p_k\rangle} p_k(x) \text{ for all natural } n.$$

10. (a) **[3]** Prove that the Gram-Schmidt process does produce orthogonal polynomials.
    (b) **[3]** Prove that any polynomial may be expressed as a sum of distinct nonzero multiples of orthogonal polynomials in precisely one way.

Solution to Problem 10:

10. (a) Consider a sequence $\{p_0, p_1, p_2, \ldots\}$ of polynomials produced by the Gram-Schmidt process. Let a polynomial p_n (n natural) be known as fully orthogonal if p_n is orthogonal to p_k for all natural $k < n$. Trivially, p_0 is fully orthogonal. Suppose that all polynomials up to n are fully orthogonal. Then, for any $m < n+1$,

$$\langle p_{n+1}, p_m\rangle = \langle x^{n+1}, p_m\rangle - \sum_{k=0}^{n} \frac{\langle x^{n+1}, p_k\rangle}{\langle p_k, p_k\rangle}\cdot\langle p_k, p_m\rangle = \langle x^{n+1}, p_m\rangle - \langle x^{n+1}, p_m\rangle = 0.$$

So by the principle of mathematical induction, any two polynomials produced by the Gram-Schmidt process are orthogonal.

(b) Note that any degree 0 (constant) polynomial may be expressed in such a way, as a multiple of the zeroth orthogonal polynomial. Suppose, for some natural n, that all polynomials of degree at most n may be expressed in such a way. Then, any polynomial $p(x)$ of degree $n+1$ is equal to $c\cdot p_n(x) + q(x)$ for some polynomial $q(x)$ of degree at most n, where $p_n(x)$ is the n^{th} orthogonal polynomial. This implies that $p(x)$ is the sum of distinct nonzero multiples of orthogonal polynomials. By the principle of mathematical induction, any polynomial may be expressed in such a way.

Suppose that there are multiple such expressions. Then, the polynomial $p(x)$ may be expressed as a sum of distinct nonzero multiples of orthogonal polynomials in more than one way. Consider the difference between two such ways. Since they are different, it is of the form $c_n p_n(x) + c_{n-1}p_{n-1}(x) + \cdots + c_0 p_0(x) = 0$ for some real numbers $c_n, c_{n-1}, \ldots, c_0$ (where $c_n \neq 0$) and natural n. The left hand side is a polynomial of degree n, since p_n has degree n, c_n is nonzero, and the other terms have degree less than n. But this cannot be equal to 0. We conclude that there must be precisely one such expression for any polynomial.

11. Let n be a nonzero natural number.

    (a) **[2]** Prove that $\langle p_n, q\rangle = 0$ for all polynomials q of degree less than n.

    (b) **[2]** Prove that if p is a nonzero polynomial that is nonnegative throughout the interval (a, b), then $\mathbf{Int}_{[a,b]}(p) > 0$. (Refer to the end of Section 1 for a relevant theorem.)

    (c) **[4]** Prove that $p_n(x)$ has precisely n distinct real roots in the interval (a, b).

Solution to Problem 11:

11. (a) By problem 10(b), we may express q as a sum of distinct nonzero multiples of orthogonal polynomials. Furthermore, the degree of q is equal to the greatest k such that p_k is present in the sum, since p_k has degree k and all other terms in the sum have degree less than k. Thus, any orthogonal polynomial present in the sum has degree strictly less than n and therefore is not p_n; their inner product must be 0. Due to the linearity of $\langle p_n, q\rangle$ in the second variable, we conclude that $\langle p_n, q\rangle = 0$.

    (b) By the Fundamental Theorem of Algebra, $p(x)$ cannot be zero for all x in the interval (a, b). Let α be a value for which $p(\alpha) \neq 0$. Let r_1 be the maximum of a and the greatest root of p less than α, and let r_2 be the minimum of b and the least root of p greater than α. Then, by the Mean Value Theorem, there exists a value c where $r_1 < c < r_2$ such that $\mathbf{Int}_{[r_1,r_2]}(p) = (r_2 - r_1)p(c)$. But $p(x)$ is positive for all x in the interval (r_1, r_2), whence $\mathbf{Int}_{[r_1,r_2]}(p) > 0$. Similarly, $\mathbf{Int}_{[a,r_1]}(p) \geq 0$ and $\mathbf{Int}_{[r_2,b]}(p) \geq 0$. Therefore, by problem 4(c),

    $$\mathbf{Int}_{[a,b]}(p) = \mathbf{Int}_{[a,r_1]}(p) + \mathbf{Int}_{[r_1,r_2]}(p) + \mathbf{Int}_{[r_2,b]}(p) > 0.$$

    (c) Let $r_1, r_2, \ldots, r_k$ be the roots of $p_n(x)$ that are real, are in the interval (a, b), and have odd multiplicity. Let $q(x) = (x - r_1)(x - r_2)\cdots(x - r_k)$, and consider $\langle p_n, q\rangle$. Note $p_n \cdot q \cdot w$ is a nonzero polynomial that is nonnegative throughout the interval (a, b). Therefore, by problem 11(b), $\langle p_n\, q\rangle > 0$. However, by problem 11(a), $\langle p_n, q\rangle = 0$ if q has degree less than n. Therefore, q must have degree n, and $k = n$. By the Fundamental Theorem of Algebra, we conclude that $p_n(x)$ must have precisely n distinct real roots in the interval (a, b).

In the study of orthogonal polynomials, two values assist in characterizing the relation between different elements of the sequence. We define sequences of these values in the following way. (Recall the functional **Lead** from Section 1 that returns the leading coefficient of a polynomial.)

$$k_n = \mathbf{Lead}(p_n) \text{ and } h_n = \langle p_n, p_n\rangle.$$

12. Let $\{p_0, p_1, p_2, \ldots\}$ be orthogonal polynomials.

    (a) **[4]** Prove, for some natural $n > 0$ and real numbers a_n and b_n independent of x, that

    $$p_{n+1}(x) - \frac{k_{n+1}}{k_n} \cdot xp_n(x) = a_n p_n(x) + b_n p_{n-1}(x).$$

    (b) **[2]** Determine the value of b_n in terms of h_{n+1}, k_{n+1}, h_n, k_n, h_{n-1}, and k_{n-1}.

Solution to Problem 12:

12. (a) Note that $p_{n+1}(x) - \frac{k_{n+1}}{k_n} \cdot xp_n(x)$ has no term of degree x^{n+1}, so its degree is at most n. By problem 10(b), there exist real numbers $c_n^{(n)}$, $c_n^{(n-1)}$, $\ldots$, $c_n^{(0)}$ such that

$$p_{n+1}(x) - \frac{k_{n+1}}{k_n} \cdot xp_n(x) = \sum_{k=0}^{n} c_n^{(k)} p_k(x).$$

If $n = 1$, we are done; otherwise, let $m < n-1$ be a natural number. Note that

$$\begin{aligned} \langle p_{n+1}, p_m \rangle - \frac{k_{n+1}}{k_n} \cdot \langle xp_n, p_m \rangle &= \sum_{k=0}^{n} c_n^{(k)} \langle p_k, p_m \rangle \\ -\frac{k_{n+1}}{k_n} \cdot \langle p_n, xp_m \rangle &= c_n^{(m)} \langle p_m, p_m \rangle. \end{aligned}$$

By problem 11(a), $\langle p_n, xp_m \rangle = 0$ since $xp_m(x)$ is a polynomial of degree less than n. Thus, $c_n^{(m)} = 0$ for all $m < n-1$. With the substitutions $a_n = c_n^{(n)}$ and $b_n = c_n^{(n-1)}$, we conclude the desired equivalence

$$p_{n+1}(x) - \frac{k_{n+1}}{k_n} \cdot xp_n(x) = a_n p_n(x) + b_n p_{n-1}(x).$$

(b) Note that $xp_{n-1}(x) = \frac{k_{n-1}}{k_n} p_n(x) + q(x)$, where q is a polynomial of degree less than n. Thus, by problem 11(a), $\langle xp_n, p_{n-1} \rangle = \langle p_n, xp_{n-1} \rangle = \frac{k_{n-1}}{k_n} \langle p_n, p_n \rangle = \frac{k_{n-1} h_n}{k_n}$.

$$\begin{aligned} \langle p_{n+1}, p_{n-1} \rangle - \frac{k_{n+1}}{k_n} \cdot \langle xp_n, p_{n-1} \rangle &= a_n \langle p_n, p_{n-1} + b_n \langle p_{n-1}, p_{n-1} \rangle \\ -\frac{k_{n+1}}{k_n} \cdot \frac{k_{n-1} h_n}{k_n} &= h_{n-1} b_n. \end{aligned}$$

We conclude that $\boxed{b_n = -\frac{k_{n-1} k_{n+1} h_n}{k_n^2 h_{n-1}}.}$

13. Answer each of the following questions in a clear and concise manner.

(a) **[4]** Is there a choice of a, b, and polynomial w that would make the sequence of polynomials $\{1, x, x^2, x^3, \ldots\}$ orthogonal with respect to the inner product described above? Why or why not?

(b) **[4]** What can you say about the relationship between the roots of two distinct orthogonal polynomials? A false response will get 0 points, while a true response will receive points in proportion to the strength of its implications.

Solution to Problem 13:

13. (a) No, there is no such choice. Because x^2 has only one distinct real root, there is no interval in which it has two distinct real roots. Therefore, by problem 11(c), the polynomial x^2 cannot be member of a sequence of orthogonal polynomials, regardless of values a and b and polynomial w.

    (b) One strong true statement is the following. False statements regarding the *precise* distribution of roots of p_n among the roots of p_k will be graded with leniency if they contain valuable content relating to the below statement.

    Let n and k be natural numbers, with $n > k$. Let $r_1, r_2, \ldots, r_k$ be the roots of p_k in increasing order. Then p_n has a root in each of the $k+1$ intervals (a, r_1), (r_1, r_2), (r_2, r_3), (r_{k-1}, r_k), $\ldots$, and (r_k, b).

    A slightly weaker statement makes the qualification $n = k + 1$.

4 Legendre Polynomials (23 pts)

One reason orthogonal polynomials are useful is that they are very good at approximating other functions. In particular, they provide a solution to the *least squares* problem for function approximation. Note our inner product is

$$\langle p, q\rangle = \mathbf{Int}_{[-1,1]}(p \cdot q) \text{ for all polynomials } p \text{ and } q.$$

Then, a **Legendre approximation** p of degree n (natural n) to the polynomial q is

$$p(x) = \sum_{k=0}^{n} \frac{\langle q, P_k\rangle}{\langle P_k, P_k\rangle} P_k(x).$$

14. (a) **[2]** Find a Legendre approximation of degree 2 to the function $f_1(x) = x^4$.

    (b) **[2]** Find a Legendre approximation of degree 3 to the function $f_2(x) = x^5$.

    (c) **[4]** Find with proof a general form for the value $\langle P_k, P_k\rangle$ in terms of natural k.

Solution to Problem 14:

14. (a) f_1 has degree 2 Legendre approximation $p_1(x) = \frac{1}{5} + \frac{4}{7}(\frac{3}{2}x^2 - \frac{1}{2}) = \boxed{\frac{6}{7}x^2 - \frac{3}{35}.}$

    (b) f_2 has degree 3 Legendre approximation $p_2(x) = \frac{3}{7}x + \frac{4}{9}(\frac{5}{2}x^3 - \frac{3}{2}x) = \boxed{\frac{10}{9}x^3 - \frac{5}{21}x.}$

    (c) We claim, for all natural k, that

$$\boxed{\langle P_k, P_k\rangle = \frac{2}{2k+1}.}$$

Note the claim holds for $k=0$, and suppose it holds for some value $k=n$. Then,

$$\begin{aligned}
\langle P_{n+1},\, P_{n+1}\rangle &= \langle P_{n+1}(x),\, \tfrac{2n+1}{n+1}xP_n(x) - \tfrac{n}{n+1}P_{n-1}(x)\rangle \\
&= \tfrac{2n+1}{n+1}\langle xP_{n+1}(x),\, P_n(x)\rangle - \tfrac{n}{n+1}\langle P_n(x),\, P_{n-1}(x)\rangle \\
&= \tfrac{2n+1}{n+1}\langle \tfrac{n+2}{2n+3}P_{n+2}(x) + \tfrac{n+1}{2n+3}P_n(x),\, P_n(x)\rangle \\
&= \tfrac{(2n+1)(n+2)}{(n+1)(2n+3)}\langle P_{n+2},\, P_n\rangle + \tfrac{2n+1}{2n+3}\langle P_n,\, P_n\rangle \\
&= \tfrac{2n+1}{2n+3}\cdot\tfrac{2}{2n+1} \text{ by our supposition} \\
&= \tfrac{2}{2(n+1)+1}.
\end{aligned}$$

By the principle of mathematical induction, we conclude that the claim holds.

15. Draw a graph of a function and one of its Legendre approximations.
    (a) **[4]** Where is the Legendre approximation a close approximation to a function? What value(s) in particular is (are) minimized by the Legendre approximation?
    (b) **[3]** Prove that the Legendre approximation of degree n of a polynomial of degree n is the polynomial itself.

Solution to Problem 15:

15. (a) Legendre approximations intersect functions they approximate many times, potentially at least as many times as the degree of the polynomial. The approximation does not reflect the slope or smoothness of the function in any way. If p is a Legendre approximation of q, then among all polynomials f of the same degree as p, the value $\mathbf{Int}_{[-1,\,1]}((f-q)^2)$ is minimized.

    (b) Note, for any natural $m \le n$, that

$$\sum_{k=0}^{n} \frac{\langle P_m,\, P_k\rangle}{\langle P_k,\, P_k\rangle} P_k(x) = \frac{\langle P_m,\, P_m\rangle}{\langle P_m,\, P_m\rangle} P_m(x) = P_m(x).$$

Thus, the degree n Legendre approximation of the m^{th} Legendre polynomial is itself. By problem 10(b), any polynomial may be expressed as a sum of distinct nonzero multiples of Legendre polynomials. Furthermore, the degree of such a polynomial is equal to the greatest n such that P_n is present in the sum, since P_n has degree n and all other terms in the sum have degree less than n. Then, by linearity of the first component of $\langle f,\, P_k\rangle$ for any natural k, we conclude that the degree n Legendre approximation of a degree n polynomial is the polynomial itself.

Just as Legendre polynomials may be defined using a recurrence relation, they may be defined with a series relation as well. For sufficiently small t (say, $|t| < \frac{1}{2x}$), the following holds for all x.

$$\sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{\sqrt{1-2xt+t^2}}.$$

16. (a) **[3]** Verify with proof the equivalence for $x=1$ and $x=-1$.
    (b) **[5]** Prove the following identity. For all natural n,

$$\frac{\sin((n+1)x)}{\sin x} = \sum_{k=0}^{n} P_k(\cos x)P_{n-k}(\cos x).$$

Solution to Problem 16:

16. (a) Let $x = 1$. Then, $P_n(x) = P_n(1) = 1$ for all n, by the definition of Legendre polynomials. Therefore, for sufficiently small t (say, $|t| < 1$),

$$\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t} = \frac{1}{\sqrt{(1-t)^2}} = \frac{1}{\sqrt{1-2t+t^2}}.$$

Now, let $x = -1$. Then, $P_n(x) = P_n(-1) = (-1)^n P_n(1) = (-1)^n$ for all n, as noted in the proof of problem 8(b). Therefore, for sufficiently small t (say, $|t| < 1$),

$$\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} (-1)^n t^n = \frac{1}{1-(-1)t} = \frac{1}{\sqrt{(1+t)^2}} = \frac{1}{\sqrt{1+2t+t^2}}.$$

(b) Recall $e^{ix} = \cos x + i \sin x$, whence the imaginary part $\Im(e^{ix}) = \sin x$. Therefore,

$$\sum_{n=0}^{\infty} e^{i(n+1)x} t^n = \frac{e^{ix}}{1-te^{ix}} = \frac{(\cos x + i\sin x)(1 - t\cos x + i \sin x)}{(1-t\cos x)^2 + (t \sin x)^2}$$

for all $|t| < 1$, from which we deduce

$$\sum_{n=0}^{\infty} \sin((n+1)x)t^n = \Im\left(\sum_{n=0}^{\infty} e^{i(n+1)x} t^n\right) = \frac{\sin x}{1 - 2\cos(x)t + t^2}.$$

It follows immediately that

$$\sum_{n=0}^{\infty} \frac{\sin((n+1)x)}{\sin x} t^n = \left(\frac{1}{\sqrt{1-2\cos(x)t+t^2}}\right)^2 = \sum_{n=0}^{\infty}\sum_{k=0}^{n} P_k(\cos x)P_{n-k}(\cos x)t^n.$$

From a term by term inspection, we conclude the identity holds for all x.