1. Isosceles trapezoid $ABCD$ has $AB = 2$, $BC = DA = \sqrt{17}$, and $CD = 4$. Point E lies on $\overline{CD}$ such that $\overline{AE}$ splits $ABCD$ into two polygons of equal area. What is DE?

   Answer: 3

   Solution: Since $ABCD$ is isosceles, dropping altitudes from A and B leaves a side length of 1 on either side of the feet of the altitudes. By the Pythagorean theorem, the trapezoid's height must be $\sqrt{17 - 1^2} = \sqrt{16} = 4$. Then the area of the trapezoid is $h \cdot \frac{AB+CD}{2} = 4 \cdot \frac{4+2}{2} = 12$, so we need $\triangle ADE$ to have area $\frac{12}{2} = 6$, which means that $DE = 2 \cdot \frac{6}{4} = \boxed{3}$.

2. At the Berkeley Sandwich Parlor, the famous BMT sandwich consists of up to five ingredients between the bread slices. These ingredients can be either bacon, mayo, or tomato, and ingredients of the same type are indistiguishable. If there must be at least one of each ingredient in the sandwich, and the order in which the ingredients are placed in the sandwich matters, how many possible ways are there to prepare a BMT sandwich?

   Answer: 192

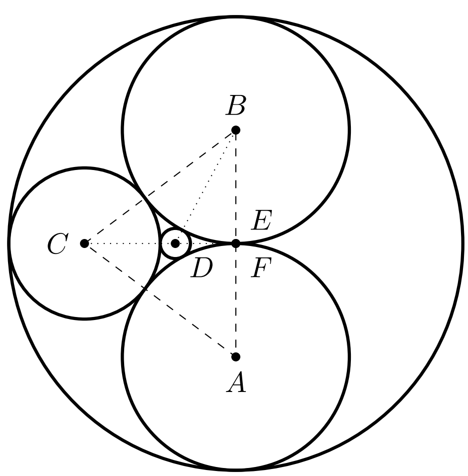
   Solution: Suppose we have exactly 5 ingredients. If there are 3 of one ingredient and 1 each of the other two, we have $\frac{5!}{3!} = 20$ ways to order them inside the sandwich, and 3 choices for the tripled ingredient, giving 60 possible sandwiches. If there are 2 each of two ingredients and only 1 of the other, then we similarly have $\frac{5!}{2!2!1!} \cdot 3 = 90$ possible sandwiches.

   For 4 ingredients, we have $\frac{4!}{2!} = 12$ orderings times 3 choices of duplicated ingredient, or 36 sandwiches (since there must always be 2 of one ingredient and 1 of the others). Finally, for 3 ingredients, we have $3! = 6$ possible orderings of 1 each of bacon, mayo, and tomato, giving $60 + 90 + 36 + 6 = \boxed{192}$ possible BMT sandwiches in total.

3. Three mutually externally tangent circles have radii 2, 3, and 3. A fourth circle, distinct from the other three circles, is tangent to all three other circles. The sum of all possible radii of the fourth circle can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

   Answer: 37

   Solution: There are two possibilities for the fourth circle. It can either be a small circle externally tangent to the three circles, or a large circle internally tangent to the three circles, as shown in the diagram.



Let r_1 be the radius of the circle such that all three circles lie outside and let r_2 be the radius of the circle such that all three circles lie inside the new circle. Let A and B be the centers of the circles with radius 3, and let C be the center of the circle with radius 2. Let D be the center of the circle with radius r_1, and let E be the center of the circle with radius r_2. Let F be the foot of the altitude of triangle $\triangle ABC$ from C.

Since CF is an altitude, by Pythagorean Theorem,

$$CF = \sqrt{CB^2 - BE^2} = \sqrt{(2+3)^2 - \left(\frac{6}{2}\right)^2} = 4$$

Note that $CF = CD + DF$, and $DF = \sqrt{BD^2 - BF^2}$, so we have:

$$2 + r_1 + \sqrt{(r_1+3)^2 - 3^2} = 4$$

Similarly, $CF = CE + EF$, and $EF = \sqrt{BE^2 - BF^2}$, so we have:

$$r_2 - 2 + \sqrt{(r_2-3)^2 - 3^2} = 4.$$

Solving the two equations above gives $r_1 = \frac{2}{5}$ and $r_2 = 6$, so the sum of the two solutions is $\frac{32}{5}$. Therefore, $m + n = \boxed{37}$.